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# Point-based methods for estimating the length of a parametric curve

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**Abstract**

This paper studies a general method for estimating the length of a parametric curve using only samples of points. We show that by making a special choice of points, namely the Gauss–Lobatto nodes, we get higher orders of approximation, similar to the behaviour of Gauss quadrature, and we derive some explicit examples.


*Keywords:* Curve; Arc length; Polynomial interpolation

## 1. Introduction

Computing the arc length of a parametric curve is a basic problem in geometric modelling and computer graphics, and has been treated in various ways. In [11], Guenter and Parent use numerical integration on the derivative of the curve. In [16], Vincent and Forsey derive a method based entirely on point evaluations. Gravesen has derived a method specifically for Bézier curves [10]. The estimation of arc length is an important issue in [13,17,18], where approximate arc length parametrizations were sought for spline curves. This is necessary, since apart from trivial cases, polynomial curves never have unit speed [6]. The article [2] treats the issue of reparametrizing NURBS curves so that the resulting curve parametrization is close to arc length. The articles [3,4] deal with optimal, i.e., as close to arc length as possible, rational reparametrizations of polynomial curves. In [15], the authors calculate approximate arc length parametrizations for general parametric curves. Recently, results have been obtained on approximating the length of a curve, given only as a sequence of points (without parameter values), using polynomials and splines [7,8].

Suppose $\mathbf{f} : [\alpha, \beta] \to \mathbb{R}^d$, $d \geqslant 2$ is a regular parametric curve, by which we mean a continuously differentiable function such that $\mathbf{f}'(t) \neq \mathbf{0}$ for all $t \in [\alpha, \beta]$, and $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$. Then its arc length (see [14, Section 9]) is

$$L(\mathbf{f}) = \int_\alpha^\beta |\mathbf{f}'(t)|\,\mathrm{d}t. \tag{1}$$

Since $L(\mathbf{f})$ is simply the integral of the 'speed' function $|\mathbf{f}'|$, a natural approach is simply to apply to $|\mathbf{f}'|$ some standard composite quadrature rule: we split the parameter interval $[\alpha, \beta]$ into small pieces, apply a quadrature rule to

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$|\mathbf{f}'|$ in each piece, and add up the contributions. If $[a, b]$ is one such piece, with $\alpha \leqslant a < b \leqslant \beta$, then a typical rule has the form

$$L(\mathbf{f}|_{[a,b]}) = \int_a^b |\mathbf{f}'(t)|\,\mathrm{d}t \approx \sum_{i=0}^{n} w_i |\mathbf{f}'(q_i)|, \tag{2}$$

for some quadrature nodes

$$a \leqslant q_0 < q_1 < \cdots < q_n \leqslant b, \tag{3}$$

and weights $w_0, w_1, \ldots, w_n$. Guenter and Parent [11] apply such a method adaptively.

This method, however, has the drawback that it involves derivatives of $\mathbf{f}$, which might be more time-consuming to evaluate than points of $\mathbf{f}$, or might simply not be available. One alternative is the 'chord length' rule (16), but it only has second order accuracy (as will be shown in 4.1). This motivated Vincent and Forsey [16] to find a higher order method using only point evaluations (18). In this paper, we investigate the following much more general point-based method, which turns out to include these two methods as special cases.

We can first interpolate $\mathbf{f}$ with a polynomial $\mathbf{p}_n : [a, b] \to \mathbb{R}^d$, of degree $\leqslant n$, at some points

$$a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b,$$

for some $n \geqslant 1$, i.e., $\mathbf{p}_n(t_i) = \mathbf{f}(t_i)$ for $i = 0, 1, \ldots, n$, giving the approximation

$$L(\mathbf{f}|_{[a,b]}) \approx L(\mathbf{p}_n|_{[a,b]}). \tag{4}$$

We can then estimate the length of $\mathbf{p}_n$ by quadrature, giving the estimate

$$L(\mathbf{p}_n|_{[a,b]}) \approx \sum_{j=0}^{m} w_j |\mathbf{p}_n'(q_j)|, \tag{5}$$

and by expressing $\mathbf{p}_n$ in the Lagrange form

$$\mathbf{p}_n(t) = \sum_{i=0}^{n} L_i(t)\mathbf{f}(t_i), \quad L_i(t) = \prod_{j=0, j\neq i}^{n} \frac{t - t_j}{t_i - t_j},$$

we get the point-based rule

$$L(\mathbf{f}|_{[a,b]}) \approx \sum_{j=0}^{m} w_j \left| \sum_{i=0}^{n} L_i'(q_j)\mathbf{f}(t_i) \right|. \tag{6}$$

In view of the definition of the length $L(\mathbf{f}|_{[a,b]})$ in (2), it is reasonable to expect that the error in (4) will be small due to the well-known fact that $\mathbf{p}_n'$ is a good approximation to $\mathbf{f}'$ when

$$h := b - a$$

is small. However, we have not seen this method explicitly referred to in the literature, nor are we aware of any error analysis. The main contribution of this paper is to offer a thorough analysis of the approximation order of the method, in terms of $h$, which depends on the points $t_i$, and the quadrature nodes and weights $q_j$ and $w_j$ as well as the smoothness of $\mathbf{f}$. One result of our analysis is that the interpolation points $t_i$ can be chosen to maximize the approximation order, analogously to the use of Gauss–Legendre points for numerical integration.

## 2. Error of the derivative-based method

For the sake of comparison, we start with a comment about the approximation order of the derivative-based method (2). If the quadrature rule used in (2) has degree of precision $r$ then the error will be of order $\mathrm{O}(h^{r+2})$ provided the $(r+1)$th derivative of $F := |\mathbf{f}'|$ is bounded [12].