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A generic formula for the values at the boundary points of monic classical orthogonal polynomials

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Abstract

In a previous paper we have determined a generic formula for the polynomial solution families of the well-known differential equation of hypergeometric type

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0.$$

In this paper, we give another such formula which enables us to present a generic formula for the values of monic classical orthogonal polynomials at their boundary points of definition.

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1. Introduction

In previous work [3], we found a generic polynomial solution for the differential equation

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0, \tag{1}$$

where $\sigma(x)=ax^2+bx+c$ is a polynomial of degree at most 2, $\tau(x)=dx+e$ is a polynomial of degree at most 1 and $\lambda_n=n(n-1)a+nd$ is the eigenvalue parameter depending on $n=0,1,2,\ldots$.

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Since we will need this formula in this article, we state it here again. In the following theorem from [3] $\bar{P}_n\left(\begin{matrix} & d & & e & \\ a & & b & & c\end{matrix}\middle| x\right)$ denotes the monic polynomial solution of Eq. (1).

2. Theorem

The main differential equation

$$(ax^2+bx+c)y_n''(x)+(dx+e)y_n'(x)-n((n-1)a+d)y_n(x)=0;\quad n\in\mathbb{Z}^+ \tag{2}$$

has a monic polynomial solution which is represented as

$$\bar{P}_n\left(\begin{matrix} & d & & e & \\ a & & b & & c\end{matrix}\middle| x\right)=\sum_{k=0}^{n}\binom{n}{k}G_k^{(n)}(a,b,c,d,e)x^k, \tag{3}$$

where

$$G_k^{(n)}=\left(\frac{2a}{b+\sqrt{b^2-4ac}}\right)^{k-n}{}_2F_1\left(\begin{matrix}k-n, & \dfrac{2ae-bd}{2a\sqrt{b^2-4ac}}+1-\dfrac{d}{2a}-n\\ & 2-d/a-2n\end{matrix}\middle|\frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right). \tag{4}$$

Note that

$${}_2F_1\left(\begin{matrix}\alpha & & \beta\\ & \gamma & \end{matrix}\middle| x\right)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}$$

is the Gauss hypergeometric function [2] and $(\alpha)_k=\Gamma(\alpha+k)/\Gamma(\alpha)$ denotes the Pochhammer symbol.

For $a=0$ these identities can be adapted by limit considerations and give (3) with

$$G_k^{(n)}(0,b,c,d,e)=\lim_{a\to 0}G_k^{(n)}(a,b,c,d,e)=\left(\frac{b}{c}\right)^{k-n}{}_2F_0\left(\begin{matrix}k-n, & \dfrac{cd-be}{b^2}+1-n\\ & -\end{matrix}\middle|\frac{b^2}{cd}\right) \tag{5}$$

which is valid for $c,d\neq 0$, leading to

$$\bar{P}_n\left(\begin{matrix} & d & & e & \\ 0 & & b & & c\end{matrix}\middle| x\right)=\left(\frac{b}{d}\right)^n\left(\frac{eb-cd}{b^2}\right)_n{}_1F_1\left(\begin{matrix}-n\\ \dfrac{eb-cd}{b^2}\end{matrix}\middle| -\frac{d}{b}x-\frac{cd}{b^2}\right). \tag{6}$$

For $a=b=0$ and $d\neq 0$ we finally get

$$\bar{P}_n\left(\begin{matrix} & d & & e & \\ 0 & & 0 & & c\end{matrix}\middle| x\right)=\lim_{\substack{a\to 0\\ b\to 0}}\bar{P}_n\left(\begin{matrix} & d & & e & \\ a & & b & & c\end{matrix}\middle| x\right)=\left(x+\frac{e}{d}\right)^n{}_2F_0\left(\begin{matrix}-\dfrac{n}{2}, & -\dfrac{n-1}{2}\\ & -\end{matrix}\middle|\frac{2cd}{(dx+e)^2}\right). \tag{7}$$

In this note, we intend to obtain another representation for the polynomial solution of the main equation (2). To reach this goal, we use the general form of the Rodrigues representation of the polynomials $\bar{P}_n\left(\begin{matrix} & d & & e & \\ a & & b & & c\end{matrix}\middle| x\right)$.