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Solutions to two functional equations arising in dynamic programming

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Abstract

This paper deals with the existence, uniqueness and iterative approximation of solutions for two functional equations arising in dynamic programming of multistage decision processes. The results presented in this paper extend, improve and unify the results due to Bellman, Bhakta and Choudhury, Bhakta and Mitra, Liu, Liu and Ume and others.

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1. Introduction

The purpose of this work is to discuss the existence of solutions for the following two functional equations arising in dynamic programming of multistage decision processes:

$$f(x) = \operatorname*{opt}_{y\in D}\{p(x, y) + A(x, y, f(a(x, y)))\}, \quad x \in S \tag{1.1}$$

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and

$$f(x)=\operatorname*{opt}_{y\in D}\{p(x,y)+q(x,y)f(a(x,y)) \\ +\operatorname{opt}\{r(x,y),s(x,y)f(b(x,y)),t(x,y)f(c(x,y))\}\},\quad x\in S, \tag{1.2}$$

where opt denotes the sup or inf, x and y stand for the state and decision vectors, respectively, a, b and c represent the transformations of the processes, and $f(x)$ denotes the optimal return function with initial state x.

Several existence and uniqueness results of solutions for some special cases of the functional equations (1.1) and (1.2) have been established in [1–7,9–12]. Bellman [2,3] was the first to investigate the existence and uniqueness of solutions for the functional equations below

$$f(x)=\inf_{y\in D}\max\{r(x,y),s(x,y)f(b(x,y))\},\quad x\in S \tag{1.3}$$

and

$$f(x)=\inf_{y\in D}\max\{r(x,y),f(b(x,y))\},\quad x\in S \tag{1.4}$$

in a complete metric space $BB(S)$. Bellman and Roosta [5] studied the iterative approximation of solutions for the functional equation

$$f(x)=\max_{y\in S(x)}\{p(x,y)+q(x,y)f(a(x,y))\}. \tag{1.5}$$

Bellman and Lee [4] pointed out that the basic form of the functional equations of dynamic programming is as follows:

$$f(x)=\sup_{y\in D}\{A(x,y,f(a(x,y)))\},\quad x\in S. \tag{1.6}$$

Bhakta and Mitra [7] obtained the existence and uniqueness of solutions for the functional equations

$$f(x)=\sup_{y\in D}\{p(x,y)+A(x,y,f(a(x,y)))\},\quad x\in S \tag{1.7}$$

in a Banach space $B(S)$ and

$$f(x)=\sup_{y\in D}\{p(x,y)+f(a(x,y))\},\quad x\in S \tag{1.8}$$

in $BB(S)$, respectively. Bhakta and Choudhury [6] established the existence of solutions for the functional equations (1.3) and (1.4) in $BB(S)$. Recently, Liu [11] and Liu and Ume [12] investigated properties of solutions for the functional equations (1.4) and

$$f(x)=\operatorname*{opt}_{y\in D}\{\alpha[p(x,y)+f(a(x,y))]+(1-\alpha)\operatorname{opt}\{r(x,y),f(a(x,y))\}\},\quad x\in S, \tag{1.9}$$

where α is a constant in [0, 1], in $BB(S)$.

This paper is divided into four sections. In Section 2, we recall some basic concepts, notations and lemmas, and establish a lemma that will be needed further on. In Section 3, we utilize the fixed-point theorem due to Boyd and Wong [8] to establish the existence, uniqueness and iterative approximation