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A construction of attracting periodic orbits for some classical third-order iterative methods

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Abstract

We use a family of root-finding iterative methods for finding roots of nonlinear equations. We present a procedure for constructing polynomials so that superattracting periodic orbits of any prescribed period occur when these methods are applied. This family includes Chebyshev's method, Halley's method, the super-Halley method, and the c-methods, as particular cases.

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1. Introduction

One of the classical problems in numerical analysis is the solution of nonlinear equations $f(x) = 0$. We may use iterative methods to approximate a solution of one of these equations. An iterative method starts from an initial guess x_0, called *pivot*, which is subsequently improved by means of an iteration

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$x_{n+1}=\Phi(x_n)$, with $n\geqslant 0$. Conditions are imposed on x_0 (and eventually on f or Φ) to ensure the convergence of the sequence $\{x_n\}_{n\geqslant 0}$ to a solution ζ, and next proceed to find the order of the convergence.

For example, Newton's iterative method $x_{n+1}=x_n-f(x_n)/f'(x_n)=N_f(x_n)$, as well as other similar second-order methods have been extensively used and studied. For polynomial equations $p(x)=0$, the iterative function N_p defines a rational mapping on the Riemann sphere. The simple roots of the equation, that is, the roots of the equation $p(x)=0$ that are not roots of the derivative $p'(x)$ are *superattracting* fixed points of N_p. In other words, if ζ is a simple root of $p(x)$, then $N_p(\zeta)=\zeta$ and $N'_p(\zeta)=0$. For the dynamics of Newton's method, see [21].

Before introducing the root-finding methods we are interested in, we recall some basic notions of complex dynamics. Let $R:\overline{\mathbb{C}}\longrightarrow\overline{\mathbb{C}}$ be a rational map on the Riemann sphere, that is, $R(x)=p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials without common factors. The degree of $R(x)$ is defined as $\deg(R)=\max\{\deg(p),\deg(q)\}$. In what follows, we will consider only rational maps of degree greater than or equal to two.

Let R be a rational map. For $x\in\overline{\mathbb{C}}$, we define its orbit as the set

$$\operatorname{orb}(x)=\{x, R(x),\ldots, R^{\circ k}(x),\ldots\},$$

where $R^{\circ k}$ stands for the k-fold iterate of R. A point x_0 is a fixed point of R if $R(x_0)=x_0$. A periodic point of period n is a point x_0 such that $R^{\circ n}(x_0)=x_0$ and $R^{\circ j}(x_0)\neq x_0$ for $0<j<n$. Observe that if $x_0\in\overline{\mathbb{C}}$ is a periodic point of period $n\geqslant 1$, then it is a fixed point of $R^{\circ n}$. A fixed point x_0 of R is *attracting*, *repelling*, or *indifferent* if $|R'(x_0)|$ is less than, greater than, or equal to 1, respectively. A *superattracting fixed point* of R is a fixed point which is also a critical point of R. A periodic point of period n is *attracting*, *superattracting*, *repelling*, or *indifferent* if it is, as a fixed point of $R^{\circ n}$, attracting, superattracting, repelling, or indifferent, respectively. The *Julia* set of a rational map R, denoted by $\mathscr{J}(R)$, is the closure of the set consisting of its repelling periodic points. Its complement is the *Fatou set*, denoted by $\mathscr{F}(R)$.

Let ζ be an attracting fixed point of R. Its *basin of attraction* is the set $B(\zeta)=\{x\in\overline{\mathbb{C}}: R^{\circ n}(x)\longrightarrow\zeta \text{ as } n\longrightarrow\infty\}$. The immediate basin of attraction of an attracting fixed point ζ of $R(x)$, denoted by $B^*(\zeta)$, is the connected component of $B(\zeta)$ containing ζ. Finally, if x_0 is an attracting periodic point of period n of R, then the basin of attraction of the orbit $\operatorname{orb}(x_0)$ is the set $B(\operatorname{orb}(x_0))=\bigcup_{j=0}^{n-1}R^{\circ j}(B(x_0))$, where $B(x_0)$ is the attraction basin of x_0 as a fixed point of $R^{\circ n}$, and its immediate basin of attraction is the set $B^*(\operatorname{orb}(x_0))=\bigcup_{j=0}^{n-1}R^{\circ j}(B^*(x_0))$. If R has an attracting periodic point x_0, then the basin of attraction is contained in the Fatou set and $\mathscr{J}(R)=\partial B(x_0)$, which is the topological boundary of $B(x_0)$. Therefore, the chaotic dynamics of R is contained in its Julia set.

For an extensive and comprehensive review of the theory of iteration of rational maps, see for example [5,19,20].

In his study on the convergence of Newton's iterative map, Cayley poses the following question: Let $p(z)$ be a polynomial. *What is the set consisting of the initial guesses* $x_0\in\mathbb{C}$ *for which the sequence of iterates* $x_n=N_p(x_{n-1})$, *with* $n\geqslant 1$, *converges to a root* ζ *of* $p(x)$. *In other words*, *what is the basin of attraction of* ζ? (See [9], as well as [10].)

We can ask the same question for an arbitrary iterative root-finding method $T_p:\overline{\mathbb{C}}\longrightarrow\overline{\mathbb{C}}$. It is clear that $\mathscr{F}(T_p)\supset\bigcup_{j=1}^{k}B(r_j)$, where $r_1,\ldots,r_k$ are the roots of $p(x)$. It is natural to ask the following questions. Let $p(x)$ be a polynomial. *What is the set consisting of the initial guesses* $x_0\in\mathbb{C}$ *for which the sequence of iterates* $x_{n+1}=T_p(x_n)$, *with* $n\geqslant 0$, *converges to a root* ζ *of* $p(x)$. *In other words*, *what is the basin of attraction of* ζ? *What is the set consisting of the points* x_0 *such that the sequence of iterates*