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Bounds on the eigenvalue range and on the field of values of non-Hermitian and indefinite finite element matrices

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Abstract

In the early seventies, Fried formulated bounds on the spectrum of assembled Hermitian positive (semi-) definite finite element matrices using the extreme eigenvalues of the element matrices. In this paper we will generalise these results by presenting bounds on the field of values, the numerical radius and on the spectrum of general, possibly complex matrices, for both the standard and the generalised problem. The bounds are cheap to compute, involving operations with element matrices only. We illustrate our results with an example from acoustics involving a complex, non-Hermitian matrix. As an application, we show how our estimates can be used to derive an upper bound on the number of iterations needed to achieve a given residual reduction in the GMRES-algorithm for solving linear systems.

Keywords: Eigenvalue bounds; Field of values; Numerical radius; Non-Hermitian finite element matrices; Iterative methods

1. Introduction

For many problems it is sufficient to know a bound on the spectrum of a matrix, without the need to know the actual eigenvalues, which may be expensive to compute. Examples are the determination of stable time steps for explicit time integration and of the iteration parameters for Chebychev-type methods. Several

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easily computable bounds on the spectrum of a matrix exist, among which the classical Gerschgorin disks are probably the best known. For a comprehensive overview of these and related bounds, we refer to [15].

In the (conforming) finite element method the discrete approximation to a partial differential operator is assembled from element matrices. Each element matrix corresponds to a local discretisation of the continuous operator. Element matrices are small in size and therefore easy to manipulate, whereas the global matrix that results from the assembly process can be very large, in which case operations with this matrix are expensive. It is therefore an attractive idea to relate the characteristics of the global matrix to the properties of the element matrices.

In the early seventies Fried studied in a number of papers [2,3] the question of how the eigenvalues of the element matrices are related to the spectrum of the global matrix. He derived simple bounds on the spectrum of the standard and generalised positive definite eigenproblem that can be computed from element eigenvalues only. Since the size of an element matrix is small the calculation of the element eigenvalues, and hence of the bounds, is a cheap operation.

His results proved to be useful tools for the analysis of the preconditioned conjugate gradient method, as exemplified by Wathen in [16,17]. In [16], Wathen shows the efficiency of a simple diagonal preconditioner for mass-matrix equations, and in [17] he analyses the element-by-element preconditioner introduced by Hughes et al. [8] for a family of Poisson problems. In [14], van Gijzen generalises these results for nonsymmetric element-by-element preconditioners in combination with GMRES. In his analysis, he determines bounds on the condition number of the preconditioned matrix in terms of the norms of element matrices. One of the drawbacks of this approach is that the convergence of GMRES is not well described by the condition number of the matrix alone.

A number of publications use the field of values of the (preconditioned) matrix to describe the convergence of GMRES, or more precisely, to give an upper bound on the reduction of the GMRES-residual norm [1,4,10,13]. The field of values is a powerful tool to study the characteristics of a (nonnormal) matrix and is for this reason a suitable means to study the convergence of GMRES for nonsymmetric matrices.

In this paper we generalise the results of Fried by providing bounds for the field of values and the numerical radius of a general matrix. Moreover, since the extension for matrix-pairs where one of the matrices is Hermitian positive definite is both natural and straightforward, we also provide bounds for this case. To illustrate our bounds, we compare them with the actual field of values and numerical radius for an example from acoustics.

Our results concern only the conforming finite element method, but do not apply to discontinuous Galerkin methods in which the global matrix is not assembled from element contributions alone, but also from inter-element jump terms.

In the last part of the paper we combine our bounds with an upper bound on the GMRES-residual norm based on the field of values. This combination allows us to make a detailed analysis of a symmetric preconditioner for a rather general family of convection-diffusion-reaction problems. Our results are surprisingly strong: we are able to derive an easily computable upper bound on the number of GMRES-iterations that is needed to obtain a given tolerance on the norm of the residual. Moreover, we are able to show for a wide range of parameters that this upper bound is mesh independent.

The structure of this paper is as follows. Section 2 provides some (well-known) background theory about assembly of finite element matrices and about the field of values and numerical radius of a matrix. Section 3 recalls the bounds of Fried on the spectrum of symmetric positive definite matrices and generalises them to bounds on the field of values and numerical range of general matrices and matrix-pairs in which