Available online at www.sciencedirect.com
SCIENCE DIRECT®
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
ELSEVIER
Journal of Computational and Applied Mathematics 189 (2006) 375–386
www.elsevier.com/locate/cam

A circular interpretation of the Euler–Maclaurin formula☆

Jean-Paul Berrut*

Department of Mathematics, University of Fribourg, CH-1700 Fribourg/Pérolles, Switzerland

Received 23 September 2004

Abstract

The present work makes the case for viewing the Euler–Maclaurin formula as an expression for the effect of a jump on the accuracy of Riemann sums on circles and draws some consequences thereof, e.g., when the integrand has several jumps. On the way we give a construction of the Bernoulli polynomials tailored to the proof of the formula and we show how extra jumps may lead to a smaller quadrature error.

MSC: 65D30

Keywords: Definite integral; Riemann sum; Trapezoidal rule; Error formula

1. Introduction

In the present work we discuss the approximation of the definite integral

$$I := \int_0^L f(x)\,\mathrm{d}x$$

of a (piecewise) smooth function f from an equidistant sample of its values by the (*composite*) *trapezoidal rule* [6–10,15]:

$$T_f(h) := h\left[\frac{f(0)}{2} + \sum_{k=1}^{N-1} f(kh) + \frac{f(L)}{2}\right], \quad h := \frac{L}{N}, \quad N \in \mathbb{N}.$$

☆ This work has been partly supported by the Swiss National Science Foundation, grant # 2000-066754/1.
* Tel.: +41 26 300 9196; fax: +41 26 300 9744.
E-mail address: jean-paul.berrut@unifr.ch.

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.02.015

The appraisal of the error $T_f(h) - I$, and the basis of one approach to Romberg extrapolation, is the standard *Euler–Maclaurin formula* (*EMF*) given in the following theorem [10]. Throughout this paper, $f^{(j)}$ will denote the *j*th derivative of f and $C^q[a, b]$ the set of all q-times continuously differentiable functions on $[a, b]$.

Theorem 1 (*EMF for the trapezoidal rule*). *Let $f \in C^{2m+2}[0, L]$ for some $m \geqslant 0$. Then, for every $N \in \mathbb{N}$ and with $h := L/N$, the error of the trapezoidal rule may be written as*

$$T_f(h) - I = a_2h^2 + a_4h^4 + \cdots + a_{2m}h^{2m} + L\,\frac{B_{2m+2}}{(2m+2)!}\,f^{(2m+2)}(\xi)h^{2m+2} \tag{1.1}$$

for some $\xi \in [0, L]$, where

$$a_{2j} := \frac{B_{2j}}{(2j)!}\,[f^{(2j-1)}(L) - f^{(2j-1)}(0)] \tag{1.2}$$

and with B_ℓ denoting the ℓth Bernoulli number.

The speed of convergence of $T_f(h)$ toward I as $h \downarrow 0$ is thus determined by the differences between the derivatives of odd orders at the extremities of the interval: in general, i.e., without the special property $f'(L) = f'(0)$, one has $\mathcal{O}(h^2)$-convergence; every equality of another odd order derivative eliminates a further h^{2j}-term. The method is therefore especially efficient when f is L-periodic and in $C^{2m+2}(-\infty, \infty)$ [16]. Notice that ξ varies with h and that the h^2-behavior of the error may show up only once h is small enough.

The question we address here is the following: *how do we understand the fact that for h small enough the integration error almost solely depends on differences in the behavior of the function at the extremities and not on what happens in-between?*

Our answer is to view the trapezoidal rule as a Riemann sum on a circle. This interpretation considers the values of f and its derivatives at the extremities as the left and right limits at a jump and explains why they govern the accuracy; it also leads to a generalization of the formula to functions with several jumps.

Note that, when the derivatives at the extremities are known, one may use them in (1.1) to construct quadrature rules with higher orders of convergence. Such rules may also be obtained without knowledge of the derivatives by replacing the latter with divided differences [2,13,14].

2. Bernoulli polynomials and Bernoulli numbers

The circle interpretation will yield as a by-product a somewhat simpler proof of the EMF. The Bernoulli polynomials (BP) are an essential ingredient of all such proofs (the Bernoulli numbers are the values of the BP at zero). They are usually described at the onset, without connection to the EMF. We shall instead construct them as recursive integrals of the constant 1 with just the right properties for a self-contained proof.

Let us first give a flavour of the latter. The trapezoidal sum is obtained from an integration by parts of $f(x) = f(x)1$ over each subinterval $[kh, (k+1)h]$, where the primitive of 1 is the line $x - c_k$ connecting