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Adaptive multiquadric collocation for boundary layer problems

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Abstract

An adaptive collocation method based upon radial basis functions is presented for the solution of singularly perturbed two-point boundary value problems. Using a multiquadric integral formulation, the second derivative of the solution is approximated by multiquadric radial basis functions. This approach is combined with a coordinate stretching technique. The required variable transformation is accomplished by a conformal mapping, an iterated sine-transformation. A new error indicator function accurately captures the regions of the interval with insufficient resolution. This indicator is used to adaptively add data centres and collocation points. The method resolves extremely thin layers accurately with fairly few basis functions. The proposed adaptive scheme is very robust, and reaches high accuracy even when parameters in our coordinate stretching technique are not chosen optimally. The effectiveness of our new method is demonstrated on two examples with boundary layers, and one example featuring an interior layer. It is shown in detail how the adaptive method refines the resolution.

Keywords: Multiquadric; Radial basis function; Integral formulation; Singular perturbations; Boundary layer problems; High-order discretizations; Spectral accuracy; Adaptive

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1. Introduction

Recently, there has been a great deal of interest in radial basis functions (RBFs) for interpolation problems, and as a tool for numerically solving differential equations. The idea of RBFs is to use linear combinations of translates of a function $\varphi(r)$ of one real variable, centred at “data centres” or “knots” x_k, to approximate an unknown function:

$$s(x) = \sum_{k=1}^{n} \lambda_k \varphi(\|x - x_k\|) + \text{low-order polynomials.} \tag{1}$$

Common choices for such functions ϕ are

Multiquadric (MQ): $\varphi(r) = \sqrt{r^2 + c^2}$,
Inverse multiquadric: $\varphi(r) = \left(r^2 + c^2\right)^{-1/2}$,
Gaussian: $\varphi(r) = \mathrm{e}^{-r^2/c^2}$.

The parameter c is the so-called shape parameter. As $c \to \infty$, the basis functions are becoming increasingly flat. The linear combination (1) can be used in an interpolation procedure, or when trying to find the solution of a differential equation.

The distinct scattered data centres x_k can be chosen arbitrarily in the domain of interest. Since RBF methods only act upon the information at the data centres, the method requires no further domain or surface integration or discretization. Hence, RBFs lead to “meshless methods”.

In [9], Franke’s numerical experiments compared 29 interpolation methods with analytic two-dimensional test functions. According to his results, the most powerful methods are the radial basis function methods based on the multiquadric basis function suggested by Hardy [12] and the thin plate spline. Madych and Nelson [20,21] showed that interpolation with the multiquadric basis is exponentially convergent. Their proof is based on reproducing kernel Hilbert spaces. Wu and Schaback [29] use a different technique to prove the same results. Their technique is general enough to handle the case of interpolation with the power spline and the thin plate spline. Since the Hilbert space is small when the radial basis function is smooth, the function being interpolated has to be extremely smooth for the error estimates to apply. Yoon [30] showed that the multiquadric basis function method converges exponentially in a Sobolov space. This was verified numerically by Fedoseyev et al. [7].

Meshless methods have been under intense scrutiny in an effort to avoid some of the problems associated with more traditional schemes. The 1990s have seen a rise in the use of meshless methods for solving partial differential equations (PDEs), led by methods from the finite element community, including the Partition of Unity Method of Babuška and Melenk [3], the h-p Cloud Method of Duarte and Oden [6], and the Element Free Galerkin Method of Belytschko et al. [4]. Motivated by the success of surface approximation, Kansa [15,16] pioneered the use of RBFs for the numerical solution of the Navier–Stokes equations. Since then, RBFs have been used to solve a variety of ordinary and PDEs.

In this paper we employ RBFs to solve boundary value problems with very thin layers. Our aim is to demonstrate that RBF methods are capable of achieving high accuracy and robustness through the introduction of adaptivity. Our solution method below is based on an integral formulation of multiquadric collocation. Integration is a smoothing operation; the convergence rate may be expected to accelerate in line with the convergence rate estimates of Madych and Nelson. Further applications of the RBF integral formulation can be found in Mai-Duy and Tran-Cong [22,23], Kansa et al. [17].