Available online at www.sciencedirect.com
SCIENCE DIRECT®
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
ELSEVIER
Journal of Computational and Applied Mathematics 185 (2006) 212–224
www.elsevier.com/locate/cam

The role of conditioning in mesh selection algorithms for first order systems of linear two point boundary value problems☆

J.R. Cash[a,*], F. Mazzia[b], N. Sumarti[a], D. Trigiante[c]

[a]Department of Mathematics, Imperial College, South Kensington, London SW7, England
[b]Dipartimento di Matematica, Università di Bari, via Orabona 4, I-70125 Bari, Italy
[c]Dipartimento di Energetica, via C. Lombroso 6/17, 50134 Firenze, Italy

Received 20 June 2003

Abstract

Codes for the numerical solution of two-point boundary value problems can now handle quite general problems in a fairly routine and reliable manner. When faced with particularly challenging equations, such as singular perturbation problems, the most efficient codes use a highly non-uniform grid in order to resolve the non-smooth parts of the solution trajectory. This grid is usually constructed using either a pointwise local error estimate defined at the grid points or else by using a local residual control. Similar error estimates are used to decide whether or not to accept a solution. Such an approach is very effective in general providing that the problem to be solved is well conditioned. However, if the problem is ill conditioned then such grid refinement algorithms may be inefficient because many iterations may be required to reach a suitable mesh on which to compute the solution. Even worse, for ill conditioned problems an inaccurate solution may be accepted even though the local error estimates may be perfectly satisfactory in that they are less than a prescribed tolerance. The primary reason for this is, of course, that for ill conditioned problems a small local error at each grid point may not produce a correspondingly small global error in the solution. In view of this it could be argued that, when solving a two-point boundary value problem in cases where we have no idea of its conditioning, we should provide an estimate of the condition number of the problem as well as the

☆The works has been supported by GNCS(INDAM) and MIUR.
*Corresponding author.
E-mail addresses: j.cash@ic.ac.uk (J.R. Cash), mazzia@dm.uniba.it (F. Mazzia).

doi:10.1016/j.cam.2005.03.007

numerical solution. In this paper we consider some algorithms for estimating the condition number of boundary value problems and show how this estimate can be used in the grid refinement algorithm.

MSC: 65L05; 65L06; 65L20

Keywords: Conditioning; Two-point boundary value problems; Mesh selection

1. Mathematical preliminaries

Many methods have been proposed for the numerical solution of nonlinear systems of first order, two-point boundary value problems of the form

$$\frac{dy}{dx} = f(x, y), \quad a \leqslant x \leqslant b, \quad g(y(a), y(b)) = 0. \tag{1}$$

Typically these methods attempt to control the error in the solution either by estimating the pointwise local error at the mesh points [7,6,14,16] or else by controlling the residual, that is the amount by which a continuous solution fails to satisfy the differential equation and the boundary conditions [9]. In both cases the codes attempt to control an estimate of the local error on the assumption that the problem is well conditioned so that if the local error in the solution is, in some sense small, then the global error will also be small. This all works perfectly well when the differential equation is well conditioned. However for stiff or ill conditioned problems, at least two major difficulties arise with such an approach. Firstly there is the problem that a small local error in the accepted solution does not necessarily mean that this solution has the required global accuracy. This can be demonstrated by a backward error analysis which shows that the error in the solution is bounded by the product of the local error and the condition number of the problem. Secondly, a mesh choosing algorithm which refines a mesh using a strategy based on estimates of the local error may do a very poor job of putting the mesh points in the correct places.

The concept of a well conditioned problem is a very intuitive one. Broadly speaking, a problem is said to be well conditioned if small changes in the functions f and g appearing in (1) produce correspondingly small changes in the solution obtained. In order to study the conditioning of (1), the standard approach is to examine the behaviour of the solutions of (1) in the neighbourhood of an isolated solution. This leads us to consider the linear equation

$$\frac{dy}{dx} = A(x)y + q(x), \quad a \leqslant x \leqslant b, \quad B_a y(a) + B_b y(b) = \beta. \tag{2}$$

If we assume that the boundary value problem (2) has a unique solution $y(x)$ then this solution is given by

$$y(x) = Y(x)Q^{-1}\beta + \int_a^b G(x,t)q(t)\,dt. \tag{3}$$

Here $Y(x)$ is a fundamental solution of (2), $G(x,t)$ is the Green’s function for (2) and the matrix Q, which is related to the boundary conditions, is defined as

$$Q = B_a Y(a) + B_b Y(b).$$