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Three-dimensional mixed finite element-finite volume approach for the solution of density-dependent flow in porous media

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Abstract

The density-dependent flow and transport problem in groundwater on three-dimensional triangulations is solved numerically by means of a mixed hybrid finite element scheme for the flow equation combined with a mixed hybrid finite element-finite volume (MHFE-FV) time-splitting-based technique for the transport equation. This procedure is analyzed and shown to be an effective tool in particular when the process is advection dominated or when density variations induce the formation of instabilities in the flow field. From a computational point of view, the most effective strategy turns out to be a combination of the MHFE and a spatially variable time-splitting technique in which the FV scheme is given by a second-order linear reconstruction based on the least-squares minimization and the Barth–Jespersen limiter. The recent saltpool problem introduced as a benchmark test for density-dependent solvers is used to verify the accuracy and reliability of this approach.

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1. Introduction

In this paper we are concerned with issues of accuracy and reliability of a numerical approach based on mixed hybrid finite element (MHFE) method for the discretization of the flow equation and a combination

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of MHFE with a high resolution finite volume (HRFV) scheme via a time-splitting technique [12] for the discretization of the transport equation, applied to three-dimensional triangulations.

Three-dimensional extension of this procedure, shown to be an effective tool for the solution of the coupled flow and transport problem in two dimensions [14,15], requires careful study. Indeed, when extending the time-splitting technique to tetrahedra, special attention has to be devoted to the choice of a truly three-dimensional limiter for the advection equation. In fact, simple generalizations of two-dimensional HRFV schemes do not preserve spatial second-order of accuracy, because of the poorly structured tetrahedra that need to be used in a three-dimensional mesh. The FV scheme that appears well suited for application to tetrahedra [16] is based on a least-squares minimization of the linear reconstruction in conjunction with the limiter proposed by Barth and Jespersen in [1].

Another difficulty that is encountered in three dimensions is given by extraordinarily long computing times to simulate some problems. This is due to extremely small time steps required by the CFL constraint for the explicit solution of the advection term. Even if different time steps are allowed for advection and dispersion in the time-splitting technique, nevertheless the small advective time steps drastically slow down the overall procedure. A strategy that considerably reduces the cost of using an explicit advection procedure while maintaining accuracy allows for spatially variable time steps. This procedure, first described and analyzed in [5] on two-dimensional rectangular meshes, is used here to obtain results in a reasonable computing time.

The saltpool problem is a benchmark problem recently introduced by Johansen et al. [10] and Oswald [18] that thoroughly tests the performance and reliability of the numerical approach. This test case is used to test the accuracy and robustness of the proposed approach and to further discuss issues related to accuracy-performance convenience of an explicit vs. an implicit FV scheme.

The paper is organized as follows. First the governing equations of the coupled flow and transport model are described. The next section is devoted to the numerical approach and in particular the spatially variable time-stepping strategy is described. The accuracy and reliability of the algorithm are then shown in the solution of the saltpool problem. Suggestions and future research directions conclude the paper.

2. The coupled flow and transport equations

The mathematical model of density-dependent flow in porous media can be expressed in terms of an equivalent freshwater head h defined as [2] $h = \psi + z$, where $\psi = p/(\rho_0 g)$ is the equivalent freshwater pressure head, p is the pressure, ρ_0 is the freshwater density, g is the gravitational constant, and z is the vertical coordinate directed upward.

The density ρ of the saltwater solution is written in terms of the reference density ρ_0 and the normalized (actual divided by maximum) salt concentration c:

$$\rho = \rho_0(1 + \varepsilon c), \tag{1}$$

where $\varepsilon = (\rho_s - \rho_0)/\rho_0$ is the density ratio, typically $\ll 1$, and ρ_s is the density of the solution at $c = 1$. The dynamic viscosity μ of the saltwater mixture is also expressed as a function of c and of the reference viscosity μ_0 as

$$\mu = \mu_0(1 + \varepsilon' c), \tag{2}$$

where $\varepsilon' = (\mu_s - \mu_0)/\mu_0$ is the viscosity ratio and μ_s is the viscosity of the solution at $c = 1$.