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Structures preserved by the QR-algorithm☆

Steven Delvaux, Marc Van Barel*

Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Leuven (Heverlee), Belgium

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Abstract

In this paper we investigate some classes of structures that are preserved by applying a (shifted) QR-step on a matrix A. We will handle two classes of such structures: the first we call polynomial structures, for example a matrix being Hermitian or Hermitian up to a rank one correction, and the second we call rank structures, which are encountered for example in all kinds of what we could call Hessenberg-like and lower semiseparable-like matrices. An advantage of our approach is that we define a structure by decomposing it as a collection of 'building stones' which we call structure blocks. This allows us to state the results in their natural, most general context.

Keywords: (shifted) QR-algorithm; Hessenberg-like matrices; Lower semiseparable-like matrices; Rank structure; Polynomial structure

1. Introduction

It is a classical result in numerical linear algebra that, when applying the QR-algorithm on an (unreduced) Hessenberg matrix, the resulting matrix is again of Hessenberg type. A similar statement can be

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* Corresponding author. Tel.: +32 16 32 75 63; fax: +32 16 32 79 46.
E-mail addresses: Steven.Delvaux@cs.kuleuven.ac.be (S. Delvaux), Marc.VanBarel@cs.kuleuven.ac.be (M.Van Barel).

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made for Hermitian matrices, thus the property of being Hermitian is also preserved by the QR-algorithm. Combining these two properties, one is led to the classical $O(n)$ algorithm for applying a QR-step on a Hermitian, tridiagonal matrix: see [7, p. 417] for the tridiagonal and [7, p. 342] for the Hessenberg case.

Besides the classical theory, recently several papers have appeared which study the QR-algorithm for specific classes of matrices, having certain structure which is preserved by applying a QR-step. In [1], several matrix shapes are studied which are invariant under the QR-algorithm. In [2], the preservation of a suitable structure is used to devise an $O(n)$ implementation for applying a QR-step on Frobenius (i.e., companion) matrices. The QR-algorithm is especially useful for this class of matrices, since it can be used here to yield an iterative solver for polynomial root location. In [5], the preservation of semiseparable plus diagonal structure (including diagonal elements) by applying a QR-step is proved as a consequence of the theory of 'rational Krylov matrices'. In [3], an $O(n)$ implementation is devised for what the authors call 'generalized semiseparable matrices'. For both these examples, the structure includes the so-called arrowhead matrices, which just as Frobenius matrices are a useful class for polynomial root location. In [10], an implicit $O(n)$ algorithm is described for applying a QR-step on a (symmetric) semiseparable matrix, using the so-called Givens-vector representation to obtain stable computations. By using a preliminary similarity transformation into semiseparable form, the QR-algorithm can be used here to compute the eigenvalue decomposition of an arbitrary Hermitian matrix: see [9].

In this paper, we investigate from a theoretical point of view two general classes of structure which are preserved by applying the QR-algorithm. The structures we consider generalize the classical and well-known cases of Hessenberg and Hermitian structures, and they also generalize the structures which are considered in the papers mentioned above (except some of the structures of [1], which are of a different flavour).

We make a distinction between two types of structure: polynomial and rank structures. For the case of rank structures, the structure can be decomposed as a collection of so-called 'structure blocks'. One feature of these structure blocks is that they are an intrinsic generalization of the 'shifted' QR-algorithm, in the sense that every block is allowed to have its own shift element. Apart from the level of generality resulting from this approach, it will also have a benefit for the proofs, which can then be restricted to the more easy case of the QR-algorithm without shift.

For the case of rank structures, in general the preservation of structure will only hold in the *nonsingular* case. The solution of the singular case is deferred to [4]; this is because it requires the introduction of several concepts (effectively eliminating QR-decompositions, sparse Givens patterns) which have a more technical flavour than the exposition in this paper, and which allow a detailed, stand-alone treatment.

For further reference, let us recall here the two defining equations of the shifted QR-algorithm. These equations show how to obtain from the matrix $A^{(v)} \in \mathbb{C}^{n\times n}$ a new iterate $A^{(v+1)}$:

$$A^{(v)} - \lambda I = QR, \tag{1}$$

$$A^{(v+1)} = RQ + \lambda I, \tag{2}$$

where $\lambda \in \mathbb{C}$ is called the *shift*, Q is unitary and R upper triangular. As it is known, by appropriately choosing shifts the matrices $A^{(v)}$ converge to (block) upper triangular form, or to diagonal form in the Hermitian case, and hence the QR-algorithm can be used to determine the eigenvalues of a given matrix $A = A^{(0)}$.

From the defining equations, one can easily deduce the following similarity relations

$$A^{(v+1)} = Q^{\mathrm{H}} A^{(v)} Q, \tag{3}$$