Available online at www.sciencedirect.com
SCIENCE DIRECT®

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER

Journal of Computational and Applied Mathematics 186 (2006) 130–147

www.elsevier.com/locate/cam

Asymmetric skew Bessel processes and their applications to finance

Marc Decamps[a,*], Marc Goovaerts[b], Wim Schoutens[c]

[a] *K.U. Leuven, FETEW, Naamsestraat 69, B-3000 Leuven, Belgium*
[b] *U.v. Amsterdam, FETEW, Naamsestraat 69, B-3000 Leuven, Belgium*
[c] *K.U. Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium*

Received 29 September 2004; received in revised form 17 January 2005

Abstract

In this paper, we extend the Harrison and Shepp's construction of the skew Brownian motion (1981) and we obtain a diffusion similar to the two-dimensional Bessel process with speed and scale densities discontinuous at one point. Natural generalizations to multi-dimensional and fractional order Bessel processes are then discussed as well as invariance properties. We call this family of diffusions *asymmetric* skew Bessel processes in opposition to skew Bessel processes as defined in Barlow et al. [On Walsh's Brownian motions, Séminaire de Probabilitiés XXIII, Lecture Notes in Mathematics, vol. 1372, Springer, Berlin, New York, 1989, pp. 275–293]. We present factorizations involving (asymmetric skew) Bessel processes with random time. Finally, applications to the valuation of perpetuities and Asian options are proposed.

Keywords: Bessel processes; Local time; Perpetuities; Asian options

1. Introduction

The *skew Brownian motion* (skew BM) was first mentioned by Itô and McKean [7]. Since then many authors have been interested in this diffusion process . We cite Walsh [16], Harrison and Shepp [6], Le Gall [9] and Ouknine [11]. A skew Brownian motion with parameter $0 \leqslant \beta \leqslant 1$ behaves like a Brownian

* Corresponding author. Tel.: +32 16 326770.
E-mail addresses: marc.decamps@econ.kuleuven.ac.be (M. Decamps), marc.goovaerts@econ.kuleuven.ac.be (M. Goovaerts), Wim.Schoutens@wis.kuleuven.ac.be (W. Schoutens).

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.03.067

motion away from the origin and is reflected to the positive side with probability β and to the negative side with probability $1-\beta$ when it hits 0. As shown in detail by Walsh [16], the resulting process is a linear diffusion with discontinuous scale and speed densities. Harrison and Shepp [6] construct the skew Brownian motion from a piecewise linear function of a time changed Brownian motion and prove, using Tanaka formula, that it is a unique strong solution to the stochastic differential equation (SDE)

$$\mathrm{d}X_\beta(t) = (2\beta-1)\,\mathrm{d}L_t^0(X_\beta) + \mathrm{d}B(t), \tag{1}$$

where $B=\{B(t), t\geqslant 0\}$ is an adapted Brownian motion and $L_t^0(X_\beta)$ is the symmetric local time of the continuous semimartingale X_β at 0. The transition density of the skew BM has a discontinuous derivative and is obtained via its Green function. The intriguing properties of the skew Brownian motion have led to applications in various disciplines. We can cite Zhang [23] in theoretical physics or Cantrell and Cosner [2] in biology.

A Bessel process of order ν, denoted by $BES^{(\nu)}$, is a linear diffusion with generator

$$\mathscr{G}^{(\nu)} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\nu+1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{2}$$

If one multiplies by x^2 the *fundamental equation* $\mathscr{G}^{(\nu)}u=\alpha u$, we recover the modified Bessel's differential equation. When $\nu=-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$ the $BES^{(\nu)}$ can be represented as the distance from the origin of a $d=(2\nu+2)$-dimensional Brownian motion. Bessel processes and some generalizations have been investigated for a long time in financial mathematics. It plays an essential role for evaluating Asian option and contingent claims under the CIR model, see Yor [22]. For an extensive survey on Bessel processes, we refer to Going–Jaeschke and Yor [5]. From a theoretical point of view, Bessel processes together with the Brownian motion are often considered as reference processes for their numerous properties. In particular the duality between the three-dimensional Bessel process and the Brownian motion killed at the origin guides to surprising results as for instance the Williams' path decomposition and the Pitman's theorem, see e.g. Yor [21] and Revuz and Yor [14].

Barlow et al. [1] construct the skew Bessel process by changing the sign with probability $1-\beta$ of the excursions away from 0 of a Bessel process of dimension $d\in(0,2)$. The symmetry of the skew Bessel processes introduced by Barlow et al. [1] has lead to interesting developments, see e.g. Watanabe [17,18].[1] In this paper, we propose an asymmetric definition of two dimensional skew Bessel process by introducing a discontinuity at some level $a\in(0,+\infty)$ of the scale and speed densities. Although the symmetry property no longer holds, we show that this family of processes shares relevant properties with the skew BM and the Bessel processes. It also provides some insight in the understanding of the exponential functionals

$$A(t) = \int_0^t \mathrm{e}^{2X_\beta(s)}\,\mathrm{d}s$$

and

$$\hat{A}(t) = \beta^2\int_0^t \mathrm{e}^{2B(s)}1_{B(s)<0}\,\mathrm{d}s + (1-\beta)^2\int_0^t \mathrm{e}^{2B(s)}1_{B(s)\geqslant 0}\,\mathrm{d}s.$$

[1] We are endebted to Marc Yor for bringing to our attention those recent papers.