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Exact and asymptotic distributions of LULU smoothers

W.J. Conradie[a,*], T. de Wet[a,1], M. Jankowitz[b]

[a] *Department of Statistics and Actuarial Science, University of Stellenbosch, Stellenbosch 7600, South Africa*
[b] *Department of Quantitative Management, University of South Africa, Pretoria 0001, South Africa*

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Abstract

This paper considers a class of non-linear smoothers, called LULU smoothers, introduced by Rohwer in the late eighties in the mathematics literature, and since then investigated fairly extensively by a number of authors for its mathematical properties. They have been successfully applied in various engineering and scientific problems. However, to date their distribution theory has not received any attention in the literature. In this paper we derive their exact as well as asymptotic distributions and show their relationship to the upper order statistics.

Keywords: Extreme value theory; Nonlinear smoothers; Asymptotic distributions

1. Introduction

Smoothing is important in many data analyses. Traditionally, linear filters were used for smoothing, especially in engineering applications where digital signal processing based on linear filters often forms part of a hardware system. However, in contrast to their behaviour for data containing well-behaved Gaussian noise, linear smoothers do not respond well to data containing impulsive noise, outliers or noise from heavy tailed distributions.

Based on his experience in the field of robustness, Tukey introduced median-based smoothers in the 1970s as a more robust smoothing technique (see [16,17]). Since then many extensions and modifications

* Corresponding author.
E-mail addresses: wjc@sun.ac.za (W.J. Conradie), tdewet@sun.ac.za (T. de Wet), jankomd@unisa.ac.za (M. Jankowitz).
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of median smoothers were proposed (see e.g., [4,10,15,19] and references contained therein). They were also generalized to more general non-linear ones based on a number of central order statistics, for example L-smoothers (see e.g. [2,3]). These median and other order statistic-based smoothers have been successful in a wide ranging spectrum of applications. See also the more general results on non-linear smoothers by, e.g., Mallows [8] and Velleman [18].

In the late 1980s and early 1990s a new class of non-linear smoothers was introduced into the literature by Rohwer [11] and Rohwer and Toerien [13]. They are based on the extreme order statistics rather than on the central ones, and Rohwer named them LULU smoothers. Since their introduction, they have been studied and applied fairly extensively, also in the engineering literature (see e.g. [7,9]). However, to date, their properties have only been studied in a deterministic setting and distribution theory based on random sequences has been lacking.

In this paper, we take a number of steps towards rectifying this situation. In the case of independent, identically distributed (i.i.d.) sequences, we derive the exact distribution of the most important members of the class of LULU smoothers. Furthermore, the limiting distributions of these smoothers as the window size increases indefinitely are derived. It is also shown how the latter relates to the limiting distributions of the upper order statistics.

The layout of the paper is as follows. In Section 2 LULU smoothers are defined and some of their mathematical properties given. In Sections 3 and 4 the exact and limiting distributions are, respectively, derived. Some numerical results are given in Section 5 and we conclude in Section 6 with a number of remarks on further work to be done.

2. LULU smoothers

Consider a doubly infinite sequence

$$s = \{\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots\}.$$

We first give a number of definitions leading to the LULU smoothers and then state (without proofs) two of their main properties.

2.1. Definitions

The building blocks of the LULU smoothers are the following two basic rank selectors: for $n = 1, 2, \ldots$ let

$$\begin{aligned}(\wedge^n s)_i &= \min(s_{i-n}, s_{i-n+1}, \ldots, s_i),\\ (\vee^n s)_i &= \max(s_i, s_{i+1}, \ldots, s_{i+n}), \quad i = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{1}$$

be the backward, minimum and forward, maximum selectors, respectively.

Remark 2.1.1. Note that $\wedge^n$ removes upward pulses and $\vee^n$ removes downward pulses.

The first step in building LULU smoothers is to combine $\wedge^n$ and $\vee^n$.

Definition 2.1.1. Define the lower and upper combinations of $\wedge^n$ and $\vee^n$ as, respectively,

$$L_n = \vee^n \wedge^n \quad \text{and} \quad U_n = \wedge^n \vee^n.$$