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Iterates of the infinitesimal generator and space–time harmonic polynomials of a Markov process

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Abstract

We relate iterates of the infinitesimal generator of a Markov process to space–time harmonic functions. First, we develop the theory for a general Markov process and create a family a space–time martingales. Next, we investigate the special class of subordinators. Combinatorics results on space–time harmonic polynomials and generalized Stirling numbers are developed and interpreted from a probabilistic point of view. Finally, we introduce the notion of pairs of subordinators in duality, investigate the implications on the associated martingales and consider some explicit examples.

Keywords: Markov process; Lévy processes; Subordinators; Space–time harmonic polynomials; Martingales; Infinitesimal generator; Stirling numbers

1. Introduction

Fundamental martingale-additive functionals can be associated to a nice Markov process X_t. There are of the type

$$M_t(f) \triangleq f(X_t) - f(X_0) - \int_0^t L_{\mathrm{e}}(f)(X_s)\,\mathrm{d}s,$$

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where L_{e} is the (extended) infinitesimal generator of X and f is any measurable function belonging to the domain of L_{e}. These martingales generate, in the Kunita–Watanabe sense, the set of all the martingales of the Markov process.

In this paper, from the martingales $M_t(f)$, we create a family of similar space–time martingales obtained by using some formulae involving the iterates of the generator. We illustrate this construction in the case of the Brownian motion and the Poisson process in Section 2 of the paper. Section 3 is devoted to the case of subordinators. Results from combinatorics (see e.g. [26]) involving space–time harmonic polynomials and generalized Stirling numbers are developed and interpreted from a probabilistic point of view. Many connections between stochastic processes and combinatorics can be found in Pitman's Saint–Flour course [20]. Relations between stochastic processes and orthogonal polynomials are described in [10], [11] and [23].

2. Iterates of the infinitesimal generator and associated martingales

2.1. Definition of the extended infinitesimal generator L_{e}

In this section, we consider a general Markov process $X=(X_t, t\geqslant 0)$ taking values in the measurable state space $(E, \mathscr{E})$ and endowed with the laws $(P_x, x\in E)$ such that

$$P_x(X_0=x)=1 \quad \text{for each } x.$$

The notion of extended (infinitesimal) generator L_{e} associated with the Markov process X was first in Kunita [14,15] and is quite convenient to exhibit important sets of martingales (under all P_x's) associated to X. More precisely,

Definition 2.1. Let f be a measurable function on E such that there exists a function $g: E \to \mathbb{R}$ and

$$M_t(f) \triangleq f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s$$

is a (P_x)-martingale for all x, then f is said to belong to D_{e}, the domain of L_{e}, the operator defined on D_{e} as

$$L_{\mathrm{e}}(f)=g.$$

Some assumptions are needed regarding the function g. In particular, g may be assumed to be bounded, but the weaker assumption

$$\int_0^t |g(X_s)|\,\mathrm{d}s<\infty \quad P_x \text{ a.s.} \quad \text{for all } x \text{ and } t.$$

is sufficient.

This definition extends that of any "stronger" infinitesimal generator L (for more details, please refer for instance, to [17–19] or [9] in Chapter XV and its errata in the last two pages of [8]). In particular, the martingale $M_t(f)$ is introduced in formula (2) p. 130 in [17].