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Algorithms for the quasiconvex feasibility problem

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Abstract

We study the behavior of subgradient projections algorithms for the quasiconvex feasibility problem of finding a point $x^* \in R^n$ that satisfies the inequalities $f_1(x^*) \leqslant 0,\ f_2(x^*) \leqslant 0, \ldots, f_m(x^*) \leqslant 0$, where all functions are continuous and quasiconvex. We consider the consistent case when the solution set is nonempty. Since the Fenchel–Moreau subdifferential might be empty we look at different notions of the subdifferential and determine their suitability for our problem. We also determine conditions on the functions, that are needed for convergence of our algorithms. The quasiconvex functions on the left-hand side of the inequalities need not be differentiable but have to satisfy a Lipschitz or a Hölder condition.

Keywords: Feasibility problem; Quasiconvex function; Normal cone; Subdifferential

1. Introduction

In this paper, we study the behavior of iterative subgradient projections algorithms for solving systems of inequalities with continuous quasiconvex functions on the left-hand side. This problem, called the *quasiconvex feasibility problem* (QFP), is defined as follows. Let R^n be the n-dimensional Euclidean space, and let $f_1(x), f_2(x), \ldots, f_m(x)$ be continuous quasiconvex functions defined on R^n. The quasiconvex feasibility problem is to find a point x^*, such that $f_1(x^*) \leqslant 0,\ f_2(x^*) \leqslant 0, \ldots, f_m(x^*) \leqslant 0$. We consider the consistent case, i.e., the case when a solution exists. The notion *quasiconvex feasibility problem* was introduced by Goffin, Luo and Ye [17], where they used cutting planes algorithms and only the differentiable case was considered there.

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The *convex feasibility problem* (CFP), which is a special case of the quasiconvex feasibility problem, was well-studied in the last decades. This fundamental problem has many applications in and outside mathematics in fields such as: optimization theory (see, e.g., [32,15,9,11]), approximation theory (see, e.g., [38,20,14]), image reconstruction from projections and computerized tomography (see, e.g., [21,22,4,6–8,10]) and other areas.

The algorithmic approach to solving the CFP was comprehensively investigated, see, e.g., [2,5], for general overviews of algorithms and, e.g., Crombez [12,13] for some recent results. In this study we investigate the possibilities of modifying and adapting some of these algorithmic schemes so that they become applicable to the QFP. In particular, we look at the *cyclic subgradient projections* (CSP) [9], *parallel subgradient projections* (PSP) [34,35] and *Eremin's algorithmic scheme* [16]. The common idea of all these algorithms is to employ projections of different types, with respect to the individual level-sets of the functions, to generate a sequence of points that converges to a solution. When the functions on the left-hand side of the inequalities are quasiconvex the situation is much more complicated because such functions lack separation properties that convex functions have. Straightforward generalizations of the aforementioned algorithms are not possible because the subdifferential of Fenchel–Moreau might be empty at some points, thus, inapplicable to quasiconvex functions.

Using different notions for subdifferentials, we develop algorithms for the QFP for functions that are not necessarily differentiable, but have to satisfy a Lipschitz or a Hölder condition. In Section 2 we present preliminary material and discuss several notions of subdifferentials. In Section 3 we present and study our algorithms for solving quasiconvex feasibility problems and clarify the relation between them and existing methods for subgradient minimization. In Section 4 we present additional algorithms for the QFP, based on a class of algorithms of Eremin.

2. Background and preliminaries

We use the books of Rockafellar [33], Hiriart-Urruty and Lemaréchal [23], as our desk-references for convex analysis. We work in the n-dimensional Euclidean space R^n where $\langle x, y\rangle$ and $\|x\|$ are the Euclidean inner product and norm, respectively. A function $f : X \to R \cup \{+\infty\}$ is a *proper function* if $\mathrm{dom}(f) := \{x \in R^n \mid f(x) < +\infty\}$ is nonempty. For any $a \in R$ the *level* (respectively, *strict level*) *set of* f, *corresponding to* a, is the set

$$\mathrm{lev}_f(a) = \{x \in R^n \mid f(x) \leqslant a\}, \tag{1}$$

respectively,

$$\mathrm{lev}^<_f(a) = \{x \in R^n \mid f(x) < a\}. \tag{2}$$

Given a set $C \subseteq R^n$, we denote by $\mathrm{int}\, C$, $\mathrm{ri}\, C$, $\mathrm{cl}\, C$ and $\mathrm{bd}\, C$ its interior, relative interior, closure and boundary, respectively.

Definition 1 (*Normal cone*). A normal cone to a set $C \subseteq R^n$ at a point $z \in R^n$ is denoted and defined by

$$N_C(z) := \{q \in R^n \mid \langle q, y - z\rangle \leqslant 0 \quad \text{for all } y \in C\}. \tag{3}$$

Observe that this definition does not require that $z \in \mathrm{cl}\, C$, see, e.g., Gromicho [19, p. 15].