J. Math. Pures Appl. 106 (2016) 995–1037

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Gaussian heat kernel bounds through elliptic Moser iteration

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A R T I C L E I N F O

Article history:
Received 6 March 2015
Available online 25 March 2016

MSC:
58J35
42B20
47D07

Keywords:
Heat kernel lower bounds
Hölder regularity of the heat semigroup
Gradient estimates
Poincaré inequalities
De Giorgi property

A B S T R A C T

On a doubling metric measure space endowed with a "carré du champ", we consider L^p estimates (G_p) of the gradient of the heat semigroup and scale-invariant L^p Poincaré inequalities (P_p). We show that the combination of (G_p) and (P_p) for $p \geq 2$ always implies two-sided Gaussian heat kernel bounds. The case $p = 2$ is a famous theorem of Saloff-Coste, of which we give a shorter proof, without parabolic Moser iteration. We also give a more direct proof of the main result in [37]. This relies in particular on a new notion of L^p Hölder regularity for a semigroup and on a characterisation of (P_2) in terms of harmonic functions.

R É S U M É

Dans un espace métrique mesuré doublant muni d'un « carré du champ », on appelle (G_p) les estimations L^p du gradient du semi-groupe de la chaleur et (P_p) les inégalités de Poincaré L^p. On montre que la combinaison de (G_p) avec (P_p) pour $p \geq 2$ implique toujours les estimations supérieures et inférieures gaussiennes du noyau de la chaleur. Le cas $p = 2$, correspondant à un résultat de Saloff-Coste, est redémontré ici avec une démonstration plus simple évitant le recours à une itération de Moser parabolique. On donne aussi une démonstration plus directe du résultat principal de [37], en utilisant en particulier une nouvelle notion de régularité Hölder L^p pour le semi-groupe et une caractérisation de (P_2) en termes de fonctions harmoniques.

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http://dx.doi.org/10.1016/j.matpur.2016.03.019

1. Introduction

1.1. The Dirichlet form setting

The reader who does not care about generality but is satisfied by a wide range of interesting examples can skip this section and in the rest of the paper think of M as a complete Riemannian manifold satisfying the volume doubling property and $\mathcal{L}$ its (nonnegative) Laplace–Beltrami operator. However, the more general setting we are about to present covers many more examples, see [35].

Let M be a locally compact separable metrisable space equipped with a Borel measure μ, finite on compact sets and strictly positive on any non-empty open set. For Ω a measurable subset of M, we shall often denote $\mu(\Omega)$ by $|\Omega|$.

Let $\mathcal{L}$ be a non-negative self-adjoint operator on $L^2(M, \mu)$ with dense domain $\mathcal{D} \subset L^2(M, \mu)$. Denote by $\mathcal{E}$ the associated quadratic form

$$\mathcal{E}(f, g) = \int_M f \mathcal{L} g \, d\mu,$$

for $f, g \in \mathcal{D}$, and by $\mathcal{F}$ its domain, which contains $\mathcal{D}$. Assume that $\mathcal{E}$ is a strongly local and regular Dirichlet form (see [28,35] for precise definitions). As a consequence, there exists an energy measure $d\Gamma$, that is a signed measure depending in a bilinear way on $f, g \in \mathcal{F}$ such that

$$\mathcal{E}(f, g) = \int_M d\Gamma(f, g)$$

for all $f, g \in \mathcal{F}$. A possible definition of $d\Gamma$ is through the formula

$$\int \varphi \, d\Gamma(f, f) = \mathcal{E}(\varphi f, f) - \frac{1}{2}\mathcal{E}(\varphi, f^2), \tag{1}$$

valid for $f \in \mathcal{F} \cap L^\infty(M, \mu)$ and $\varphi \in \mathcal{F} \cap \mathcal{C}_0(M)$. Here $\mathcal{C}_0(M)$ denotes the space of continuous functions on M that vanish at infinity. According to the Beurling-Deny–Le Jan formula, the energy measure satisfies a Leibniz rule, namely

$$d\Gamma(fg, h) = f d\Gamma(g, h) + g d\Gamma(f, h), \tag{2}$$

for all $f, g \in \mathcal{F} \cap L^\infty(M, \mu)$ and $h \in \mathcal{F}$, see [28, Section 3.2]. One can define a pseudo-distance d associated with $\mathcal{E}$ by

$$d(x, y) := \sup\{f(x) - f(y); f \in \mathcal{F} \cap \mathcal{C}_0(M) \text{ s.t. } d\Gamma(f, f) \leq d\mu\}. \tag{3}$$

Throughout the whole paper, we assume that the pseudo-distance d separates points, is finite everywhere, continuous and defines the initial topology of M (see [56] and [35, Subsection 2.2.3] for details).

When we are in the above situation, we shall say that $(M, d, \mu, \mathcal{E})$ is a metric measure (strongly local and regular) Dirichlet space. Note that this terminology is slightly abusive, in the sense that in the above presentation d follows from $\mathcal{E}$.

For all $x \in M$ and all $r > 0$, denote by $B(x, r)$ the open ball for the metric d with centre x and radius r, and by $V(x, r)$ its measure $|B(x, r)|$. For a ball B of radius r and $\lambda > 0$, denote by λB the ball concentric with B and with radius λr. We sometimes denote by $r(B)$ the radius of the ball B. Finally, we will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \leq Cv$ and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$.