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Weakly regular fluid flows with bounded variation on the domain of outer communication of a Schwarzschild black hole spacetime

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ABSTRACT

We study the global dynamics of isothermal fluids evolving in the domain of outer communication of a Schwarzschild black hole. We first formulate the initial value problem within a class of weak solutions with bounded variation (BV), possibly containing shock waves. We then introduce a version of the random choice method and establish a global-in-time existence theory for the initial value problem within the proposed class of weakly regular fluid flows. The initial data may have arbitrary large bounded variation and can possibly blow up near the horizon of the black hole. Furthermore, we study the class of possibly discontinuous, equilibrium solutions and design a version of the random choice method in which these fluid equilibria are exactly preserved. This leads us to a nonlinear stability property for fluid equilibria under small perturbations with bounded variation. Furthermore, we can also encompass several limiting regimes (stiff matter, non-relativistic flows, extremal black hole) by letting the physical parameters (mass of the black hole, light speed, sound speed) reach extremal values.

RÉSUMÉ

On étudie la dynamique globale d'un fluide isotherme, évoluant dans le domaine de communication exterieur d'un espace-temps de Schwarzschild. On formule le problème de Cauchy dans la classe des solutions à variation bornée contenant des ondes de choc. On propose ensuite une version de la méthode de Glimm et on démontre un théorème d'existence globale en temps pour les écoulements de fluides faiblement réguliers. La donnée initiale peut avoir une grande variation totale et n'est pas nécessairement bornée près de l'horizon du trou noir de Schwarzschild. De plus, on étudie la classe des solutions stationnaires (éventuellement discontinues) de ce problème et on propose une version de la méthode de Glimm qui préserve ces équilibres. Ceci nous conduit à la stabilité nonlinéaire de ces solutions stationnaires sous des perturbations dont la variation totale reste petite. Enfin, on considère plusieurs cas limites (matière « rigide », limite non-relativiste, trou noir extrême),

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qui sont obtenus en faisant tendre les paramètres physiques (masse du trou noir, vitesse de la lumière, vitesse du son) vers leurs valeurs extrêmes.

1. Introduction

We are interested in compressible fluids evolving on a curved background and, specifically, on the domain of outer communication of a Schwarzschild black hole spacetime. The fluid flows under consideration may contain shock waves and we must work within a class of weak solutions to the Euler equations. Our main result in this paper is a global-in-time existence theory for the initial value problem, when the fluid data are prescribed on a spacelike hypersurface. We also establish the nonlinear stability of equilibrium fluid solutions and investigate various limiting regimes when the light speed denoted by $c \in (0, +\infty)$, the (constant) sound speed denoted by $k \in [0, +\infty)$, and the mass of the back hole denoted by $M \in [0, +\infty)$ reach extremal values.

Recall that Schwarzschild spacetime is a spherically symmetric[1] solution to the vacuum Einstein equations of general relativity, and describes a massive body surrounded by a vacuum region. It is one of the simplest non-flat solution to the Einstein equations, but yet the analysis of (linear and) nonlinear waves propagating on this spacetime is very challenging and has attracted a lot of attention by mathematicians in recent years. The present paper is part of a program initiated by the first author on the Cauchy problem for the Einstein–Euler equations: see [1,2,9,14–16], as well as the graduate course [13] on self-gravitating matter and weakly regular spacetimes.

In the so-called Schwarzschild coordinates $t \geq 0$ and $r \in (2M, +\infty)$, the domain of outer communication of Schwarzschild spacetime is described by the metric

$$g = -\left(1 - \frac{2M}{r}\right)c^2\, dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, g_{S^2}, \qquad r > 2M, \tag{1.1}$$

in which $g_{S^2} := d\theta^2 + (\sin\theta)^2\, d\varphi^2$ is the canonical metric on the two-sphere S^2, with $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$. Observe that the metric coefficients are singular as $r \to 2M$, but this boundary is not a genuine singularity of the spacetime and the coefficients would become regular at $r = 2M$ by suitably changing coordinates and the metric could be extended beyond this boundary. The boundary $r = 2M$ is the horizon of the black hole, and it is natural to study the dynamics of nonlinear waves outside the black hole region.

The Levi-Civita connection associated with (1.1) being denoted by ∇, the Euler equations for a perfect compressible fluid on this spacetime read

$$\nabla_\alpha\big(T^\alpha_\beta(\rho, u)\big) = 0, \tag{1.2}$$

in which the energy–momentum tensor

$$T^\alpha_\beta(\rho, u) = \rho c^2 u^\alpha u_\beta + p(\rho)\Big(g^\alpha_\beta + u^\alpha u_\beta\Big) \tag{1.3}$$

(with $c > 0$ denoting the speed of light) depends on the mass-energy density of the fluid $\rho : M \mapsto (0, +\infty)$ and its velocity field $u = (u^\alpha)$, normalized to be unit and future oriented:

$$u^\alpha u_\alpha = -1, \qquad u^0 > 0. \tag{1.4}$$

[1] That is, invariant under the group of rotations $SO(3)$.