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On a prescribed mean curvature equation in Lorentz–Minkowski space

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ABSTRACT

We are interested in providing new results on the following prescribed mean curvature equation in Lorentz–Minkowski space

$$\nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + u^p = 0,$$

set in the whole $\mathbb{R}^N$, with $N \geqslant 3$.
We study both existence and multiplicity of radial ground state solutions (namely positive and vanishing at infinity) for $p > 1$, emphasizing the fundamental difference between the subcritical and the supercritical case.
We also study speed decay at infinity of ground states, and give some decay estimates.
Finally we provide a multiplicity result on the existence of sign-changing bound state solutions for any $p > 1$.

RÉSUMÉ

On montre de nouveaux résultats sur l'équation de courbure moyenne prescrite dans l'espace de Lorentz–Minkowski

$$\nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + u^p = 0,$$

dan le plein $\mathbb{R}^N$, avec $N \geqslant 3$.
On étudie à la fois l'existence et la multiplicité des états fondamentaux à symétrie sphérique (solutions positives et qui s'annulent à l'infini) pour $p > 1$, soulignant la différence fondamentale entre le cas sous-critique et le cas supercritique.
On étudie aussi la vitesse de la décroissance à l'infini des états fondamentaux et on montre une estimation de cette décroissance.

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Enfin, on démontre un résultat de multiplicité pour les solutions qui s'annulent à l'infini et qui changent signe pour tous $p > 1$.

0. Introduction

In this paper we are mainly interested in finding radial solutions for the problem

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right] + u^p = 0, \\ u(x) > 0, \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{$\mathcal{P}_+$}$$

where $N \geqslant 3$ and $1 < p$.

The equation at the first line is quasilinear and involves the so called mean curvature operator in the Lorentz–Minkowski space which has been object of investigation in some recent papers.

The Euclidean version of the problem, where our equation is replaced by

$$\nabla \cdot \left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right] + u^p = 0,$$

has been studied for example by Ni and Serrin [19] and del Pino and Guerra [15] (see also the references therein). In those papers multiplicity and non existence results have been proved, depending on the choice of p.

At present, the literature concerning our equation is quite poor and mainly focused on the problem of finding positive solutions satisfying Dirichlet boundary conditions in bounded domains (see [3–5,11–13]). In unbounded domains, and in particular in the whole $\mathbb{R}^N$, equations involving mean curvature operator in Lorentz–Minkowski space are almost unexplored even if they have a considerable appeal from both physical and mathematical point of view (we refer to [14] and the references therein).

In particular, we recall the strict relation between the equation we treat and the Born–Infeld (B–I for short) model in the theory of nonlinear electrodynamics. Assuming, in a static setting, that the magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$ is everywhere null and expressing the electric field as $\mathbf{E} = -\nabla u$, the B–I Lagrangian displays

$$\mathcal{L} = b^2 \left(1 - \sqrt{1 - \frac{|\nabla u|^2}{b^2}}\right),$$

and the corresponding Euler–Lagrange equation is $\nabla \cdot \left[\frac{\nabla u}{\sqrt{b^2-|\nabla u|^2}}\right] = 0$. In the same spirit of [9], Benci and Fortunato [2] proposed to describe the charged particles electrodynamics replacing B–I Lagrangian with the Maxwell one, and preserving the nonlinear structure by adding a perturbation $W(\sigma)$, where $\sigma = |\mathbf{A}|^2 - |u|^2$ is a Poincaré invariant which makes the theory they developed consistent with general relativity.

Even if our equation is, in some sense, the effect of a sort of combination of the two theories, since it arises perturbing the electrostatic B–I Lagrangian with a pure power nonlinearity (we just assume $b = 1$ for convenience), we remark that our study does not pursue the same physical purpose as [2] and [9]. Indeed, as observed in [2], solutions of problem $(\mathcal{P}_+)$ have negative energy and then they are not suitable to represent charged particles (in general relativity energy corresponds to mass and then it must be positive).

Our study aims to add some new results to the work by Bonheure, De Coster and Derlet [8], where problem $(\mathcal{P}_+)$ was firstly studied. There they proved that if $p > \frac{N+2}{N-2} := 2^* - 1$ (the so called supercritical