J. Math. Pures Appl. 106 (2016) 1141–1173

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Stochastic regularization effects of semi-martingales on random functions

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ARTICLE INFO

Article history:
Received 21 July 2015
Available online 19 April 2016

MSC:
primary 35R60, 60H07
secondary 35Q84

Keywords:
Stochastic regularization
(Backward) stochastic partial differential equations
Malliavin calculus

ABSTRACT

In this paper we address an open question formulated in [16]. That is, we extend the Itô–Tanaka trick, which links the time-average of a deterministic function f depending on a stochastic process X and F the solution of the Fokker–Planck equation associated to X, to random mappings f. To this end we provide new results on a class of adapted and non-adapted Fokker–Planck SPDEs and BSPDEs.

RÉSUMÉ

Dans cet article, on étend la méthode dite « Itô–Tanaka trick » qui relie l'intégrale en temps d'une fonction déterministe f le long d'un processus stochastique X à la solution F de l'équation de Fokker–Planck associée à X, au cas où f dépend de l'aléa sous-jacent au modèle. Cette question constitue un problème ouvert posé dans la référence [16]. Notre approche utilise de nouveaux résultats concernant une classe d'EDPS de Fokker–Planck rétrogrades adaptées et non adaptées.

1. Introduction

In [16], the authors analyzed the effects of a multiplicative stochastic perturbation on the well-posedness of a linear transport equation. One of the key tool in their analysis is the so-called *Itô–Tanaka trick* which links the time-average of a function f depending on a stochastic process and F the solution of the Fokker–Planck equation associated to the stochastic process. More precisely, the formula reads as

$$\int_0^T f(t, X_t^x)dt = -F(0, x) - \int_0^T \nabla F(t, X_t^x) \cdot dW_t, \ \mathbb{P} - a.s., \tag{1.1}$$

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[1] The author is grateful to *IDEX de Toulouse* for financial support.

http://dx.doi.org/10.1016/j.matpur.2016.04.004

where $(X_t^x)_{t\geq 0}$ is a solution of the stochastic differential equation

$$X_t^x = x + \int_0^t b(s, X_s^x) ds + W_t, \tag{1.2}$$

and F is the solution of the backward Fokker–Planck equation

$$F(t,x) = \int_t^T \left(\frac{1}{2}\Delta + b(s,x)\cdot\nabla\right) F(s,x) ds - \int_t^T f(s,x) ds. \tag{1.3}$$

In [24], by means of suitable regularity results for solutions of parabolic equations in $L^q(L^p)$ spaces, the authors showed, assuming $f, b \in E := L^q([0,T]; L^p(\mathbb{R}^d))$ with $2/q + d/p < 1$, that $F \in L^q([0,T]; W^{2,p}(\mathbb{R}^d))$. Hence, in the weak sense, F has 2 additional degrees of regularity compared to f in E. Thus, formula (1.1) tells us that the time-average of f with respect to the stochastic process $(X_t^x)_{t\geq 0}$ is more regular than f itself (it has 1 additional degree of regularity). This is what we call *a stochastic regularization effect* or *regularization by noise*. In this paper, we investigate the following open question stated in [16]:

"The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem. Specifically there are already some difficulties in dealing with a vector field b which depends itself on the random perturbation W. There is no obvious extension of the Itô–Tanaka trick to integrals of the form $\int_0^T f(\omega, s, X_s^x(\omega)) ds$ *with random f."*

A major "pathology" in the framework of stochastic regularization is the existence of random functions f for which the Itô–Tanaka trick should not improve the regularity of f. For instance, in [16], the authors consider a random function $\tilde{f}$ of the form

$$\tilde{f}(\omega, s, x) := f(x - W_s(\omega)),$$

where $(W_t)_{t\geq 0}$ is the Brownian motion from (1.2). This gives, for $b = 0$ in (1.2),

$$\int_0^T \tilde{f}(\omega, t, W_t + x) dt = \int_0^T f(t,x) dt$$

which does not bring any additional regularity. It turns out that, when f is a random function, the solution F to (1.3) is not adapted anymore to $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ the natural filtration of the Brownian motion, making the stochastic integral on the right-hand side of (1.1) ill-posed.

In this paper we tackle this difficulty by considering another equation which is the *adapted* version of the Fokker–Planck equation (1.3). More precisely, we show in Theorem 3.2 that given random functions b and f which depend in an adapted way, of a standard Brownian motion $(W_t)_{t\geq 0}$, the following formula holds:

$$\int_0^T f(t, X_t^x) dt = -F(0,x) - \int_0^T \left(\nabla F(s, X_s^x) + Z(s, X_s^x)\right) dW_s - \int_0^T \mathrm{div} Z(s, X_s^x) ds, \ \mathbb{P} - a.s., \tag{1.4}$$

where (F, Z) is the *adapted* mild solution of the following backward stochastic partial differential equation (BSPDE)

$$F(t,x) = \int_t^T \left(\frac{1}{2}\Delta + b(s,x)\cdot\nabla\right) F(s,x) ds - \int_t^T f(s,x) ds - \int_t^T Z(s,x) dW_s, \tag{1.5}$$