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On the Closing Lemma for planar piecewise smooth vector fields

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ABSTRACT

A large number of papers deal with "Closing Lemmas" for C^r-vector fields (and C^r-diffeomorphisms). Here, we introduce this subject and formalize the terminology about nontrivially recurrent points and nonwandering points for the context of planar piecewise smooth vector fields. A global bifurcation analysis of a special family of piecewise smooth vector fields presenting a nonwandering set with nontrivial recurrence is performed. As a consequence, we are able to say that the Classical and the Improved Closing Lemmas are false for this scenario.

RÉSUMÉ

Plusieurs articles traitent le problème du lemme de fermeture pour des champs de vecteurs C^r (et des C^r-difféomorphismes). Dans cet article, on introduit ce problème et on formalise la terminologie des points récurrents non triviaux et, également celle des points non errants dans le contexte des champs de vecteurs planaires de classe C^r par morceaux. On fait une analyse globale des bifurcations d'une famille de champs de vecteurs de classe C^r par morceaux possédant un ensemble récurrent et non errant. Comme conséquence, on en déduit que le lemme de fermeture est faux dans notre contexte.

1. Introduction

One of the most challenging problems in the theory of dynamical systems (of continuous or discrete time) is the so-called **Closing Lemma** (and variations thereof). Roughly speaking, in this problem the system has a nonperiodic point x_0 and the trajectory by x_0 return to a small neighborhood of x_0 infinitely many times. The objective is to obtain a small perturbation of the original system in such a way that the new system has a closed trajectory through x_0.

This is an old problem (probably first stated by Poincaré in [16], vol. 1, p. 82 in 1899) that has a lot of contributions along the history. This question is so relevant that Smale (in [22]) established it as one of the

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problems of the XXI century (in fact, it is the Problem 10). We strongly suggest reading the survey [1] about references and the main ideas behind this theme. Several versions of "Closing Lemmas" were stated with slightly distinct hypothesis and conclusions. As a consequence several proofs were made (see [13–15,17–19]) for these special cases and several cases where "Closing Lemmas" fail (see [10,11,20]) were stated.

A Piecewise Smooth Vector Field (PSVF, for short) $Z = (X, Y)$ on the plane is a pair of C^r-vector fields X and Y, where X and Y are restricted to regions of the plane separated by a smooth curve Σ (see the books [2] and [21] and references therein). In the context of PSVFs, a formal theoretical approach about minimal sets and chaos is in the beginning and only papers like [3,4,8,12] address the issue. As far as the author knows, there are no papers dealing with any version of the Closing Lemma for PSVFs. So, here we introduce this investigation and establish some terminology. Specifically, we stated Definitions 5 and 6 that formalize the concepts of nontrivially recurrent points, nonwandering points and distinguish two kinds of periodic points.

We also consider perturbations of a PSVF presenting a nonwandering set with nontrivial recurrence. All topological types in a neighborhood of such system are considered. The observation of such topological types, achieved by means of parametric piecewise smooth perturbations of the model, reveals that no one of them presents periodic points with the desired properties. As we shall explain below, in order to obtain such periodic points, even the traditional approach considering a local C^r-perturbation of some nonwandering/nontrivially recurrent point, known as a *C^r-surgery* (see Subsection 4.2.1) is unsuccessful.

The paper is organized as follows: In Section 2 we give a brief introduction to the PSVFs theory. In Section 3 we state the main results of the paper. In Section 4 we prove the main results. In Section 5 we give some conclusions about the paper.

2. Preliminaries

Now we formalize some basic concepts about PSVFs that pave the way in order to announce the main results. Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$ and consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. Observe that we can assume, locally around the origin of $\mathbb{R}^2$, that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for } (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for } (x,y) \in \Sigma^-, \end{cases} \tag{1}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \chi$. Consider on Ω the product topology. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [9], that we summarize next. Indeed, consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2,$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c.
(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.
(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.