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Flag bundles on Fano manifolds

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A B S T R A C T

As an application of a recent characterization of complete flag manifolds as Fano manifolds having only $\mathbb{P}^1$-bundles as elementary contractions, we consider here the case of a Fano manifold X of Picard number one supporting an unsplit family of rational curves whose subfamilies parameterizing curves through a fixed point are rational homogeneous, and we prove that X is homogeneous. In order to do this, we first study minimal sections on flag bundles over the projective line, and discuss how Grothendieck's theorem on principal bundles allows us to describe a flag bundle upon some special sections.

R É S U M É

Un résultat récent identifie les variétés de drapeaux complets comme les variétés de Fano n'ayant que des fibrations en $\mathbb{P}^1$ comme contractions élémentaires. On va se servir de cette caractérisation pour étudier une variété de Fano X de nombre de Picard un admettant une famille non scindée de courbes rationnelles dont les sous-familles qui paramétrisent les courbes par un point choisi sont rationnellement homogènes. On montre qu'un tel X est homogène. Dans ce but, on étudie d'abord les sections minimales de fibrés en drapeaux sur la droite projective et on discute comment le théorème de Grothendieck sur des fibrés principaux permet de décrire un fibré en drapeaux à partir de certaines sections spéciales.

1. Introduction

Although vector bundles and their projectivizations play an important role in every branch of Algebraic Geometry, a framework in which these objects are especially manageable is the one of algebraic varieties containing rational curves. The main reason for this is the fact, attributed to C. Segre for rank two (see [20, p. 44] for historical remarks on the general statement), that every vector bundle over the projective

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line $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles. However, it was not until Alexandre Grothendieck's celebrated paper [6] that this theorem achieved an optimal form, in the framework of principal bundles. In fact, Grothendieck shows (by reducing the general case, via the adjoint representation of G, to the study of orthogonal bundles over $\mathbb{P}^1$) that every principal G-bundle – for G reductive – is diagonalizable (see Theorem 3.3 below).

Our interest in this topic comes from its relation with certain homogeneity criteria that we have been recently considering ([17,19]) within a project whose goal is the Campana–Peternell conjecture, which predicts that every Fano manifold with nef tangent bundle is rational homogeneous. In a nutshell, we showed that flag manifolds are characterized within the class of Fano manifolds by having only $\mathbb{P}^1$-bundles as elementary contractions. In particular, one may then try to use this result to prove the homogeneity of a certain Fano manifold X by "untangling" its families of extremal rational curves, constructing upon X another Fano manifold $\widetilde{X}$ dominating it and satisfying the above property.

More concretely, this "bottom-up" strategy may be roughly described as follows: we start from a Fano manifold whose homogeneity we want to check; we consider a (not necessarily complete) proper dominating family of minimal rational curves in X, $\mathcal{M} \xleftarrow{p} \mathcal{U} \xrightarrow{q} X$, and ask ourselves whether $\widetilde{X} = \mathcal{U}$ is again a Fano manifold and, in this case, proceed by substituting X by $\widetilde{X}$. If this procedure can be carried out until we get to a Fano manifold in which all the families of minimal rational curves are $\mathbb{P}^1$-bundles, then the original variety X will be homogeneous.

This process can be shortened in the particular case in which q is smooth and its fibers $\mathcal{M}_x := q^{-1}(x)$ are homogeneous, since rational homogeneous bundles are determined by principal bundles, leading us immediately to a complete flag bundle $\widetilde{\mathcal{U}}$ dominating $\mathcal{U}$ (see Section 2 below). In order to check that $\widetilde{\mathcal{U}}$ is in fact a complete flag, we need to study sections of the bundle $\widetilde{\mathcal{U}}$ over minimal rational curves in X, which is our motivation to give a "rational curves oriented" interpretation of Grothendieck's theorem.

Section 3 is devoted to this topic. More concretely, we will study G/B-bundles (with G semisimple and $B \subset G$ a Borel subgroup) over the Riemann sphere $\mathbb{P}^1$. We will see that (up to a choice of a Cartan subgroup $H \subset B$) in each of these bundles we may define a set of sections, that we call *fundamental*, that are in one-to-one correspondence with the Weyl group of G and that, under this correspondence, reflections of the root system correspond to $\mathbb{P}^1$-bundles containing pairs of these sections. Moreover, we will show that one of these sections is minimal – in a deformation theoretical sense, see Definition 3.1 – and that the G/B-bundle is determined by this minimal section and by its self-intersection numbers within the $\mathrm{rk}(G)$ $\mathbb{P}^1$-bundles containing it. This information may be then represented by what we call a *tagged Dynkin diagram* (see Theorem 3.23 for a precise statement).

As an application, we prove, in Section 4, the following statement:

Theorem 1.1. *Let X be a Fano manifold of Picard number one, and $p : \mathcal{U} \to \mathcal{M}$ be an unsplit dominating complete family of rational curves satisfying that the evaluation morphism $q : \mathcal{U} \to X$ is smooth. Assume furthermore that the fiber $\mathcal{M}_x = q^{-1}(x)$ is a rational homogeneous space for every $x \in X$. Then X is rational homogeneous.*

The relation of this statement with the Campana–Peternell conjecture comes from the fact that the smoothness of q is satisfied by any unsplit family of rational curves in X if we assume that T_X is nef.

We note also that this results resembles the main theorem in [7] (proven first by Mok for Hermitian symmetric spaces and homogeneous contact manifolds, see [16]), but there are certain differences between the two statements: on one hand Hwang and Hong need to assume that the image of $\mathcal{M}_x$ into $\mathbb{P}(\Omega_{X,x})$, the so called *VMRT of $\mathcal{M}$ at x*, is projectively equivalent to the VMRT of a rational homogeneous manifold X', while we do not need to consider any particular projective embedding of $\mathcal{M}_x$. On the other hand, they only need to check the above property on a general point $x \in X$, while we need to assume that q is smooth,