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Equivalent formulations for the branched transport and urban planning problems

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ABSTRACT

We consider two variational models for transport networks, an *urban planning* and a *branched transport* model, in both of which there is a preference for networks that collect and transport lots of mass together rather than transporting all mass particles independently. The strength of this preference determines the ramification patterns and the degree of complexity of optimal networks. Traditionally, the models are formulated in very different ways, via cost functionals of the network in case of urban planning or via cost functionals of irrigation patterns or of mass fluxes in case of branched transport. We show here that actually both models can be described by all three types of formulations; in particular, the urban planning can be cast into a Eulerian (*flux-based*) or a Lagrangian (*pattern-based*) framework.

RÉSUMÉ

On étudie deux modèles variationnels de transport par des réseaux, le modèle de *planification urbaine* et le modèle de *transport branché*. Les deux modèles favorisent des réseaux qui regroupent et transportent beaucoup de masse simultanément au lieu de transporter les particules individuellement. La ramification et la complexité des réseaux optimaux sont déterminés par le degré de cette préférence. Traditionnellement, les formulations des deux modèles diffèrent beaucoup. Une formulation utilise une fonction-coût sur le réseau (dans le cas de la planification urbaine), l'autre une fonction-coût sur les systèmes d'irrigation ou les flux matériels (dans le cas du transport branché). On démontre que les deux modèles permettent en fait les trois formulations ; en particulier le problème de la planification urbaine peut être formulé dans un contexte eulérien (par les *flux*) ou lagrangien (par les *systèmes d'irrigation*).

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1. Introduction

The object of this paper are two transport problems, namely the *branched transport problem* and the *urban planning problem*. In general terms, a transport problem asks how to move mass from a given initial spatial distribution to a specific desired final mass distribution at the lowest possible cost. Different cost functionals now lead to quite different optimisation problems.

Monge's problem The prototype of all these problem is Monge's problem. In a rather general setting, let μ_+, μ_- be finite positive Borel measures on $\mathbb{R}^n$ with the same total mass. The transportation of μ_+ onto μ_- is modelled by a measurable map $t : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu_-(B) = \mu_+(t^{-1}(B))$ for all Borel sets B. The cost to move an infinitesimal mass $\mathrm{d}\mu_+(x)$ in the point x to the point $t(x)$ is given by $d(x, t(x))\,\mathrm{d}\mu_+(x)$, and the total cost is then given by the formula

$$\int_{\mathbb{R}^n} d(x, t(x))\,\mathrm{d}\mu_+(x)\,. \tag{1.1}$$

Usually, $d(x, y) = |x - y|^p$ for some $p \geq 1$.

Branched transport problem Monge's cost functional is linear in the transported mass $\mathrm{d}\mu_+(x)$ and thus does not penalise spread out particle movement. Each particle is allowed to travel independently of the others. This feature makes the Monge's problem unable to model systems which naturally show ramifications (e.g., root systems, leaves, the cardiovascular or bronchial system, etc.). For this reason, the *branched transport problem* has been introduced by Maddalena, Morel, and Solimini in [1] and by Xia in [2]. It involves a functional which forces the mass to be gathered as much as possible during the transportation. This is achieved using a cost which is strictly subadditive in the moved mass so that the marginal transport cost per mass decreases the more mass is transported together (cf. Fig. 1). We will formally introduce branched transport later in Section 1.1.

Urban planning problem The second problem we are interested in is the *urban planning problem*, introduced in [3]. Here, the measures μ_+, μ_- have the interpretation of the population and workplace densities, respectively. In this case the cost depends on the public transportation network Σ, which is the object of optimisation. In fact, one part of the cost associated with a network Σ is the optimal value of (1.1), where the cost d depends on Σ and is chosen in such a way that transportation on the network is cheaper than outside the network. The other part models the cost for building and maintaining the network. A detailed, rigorous description is given in Section 1.2.

Patterns and graphs Branched transport has been studied extensively and has several formulations. Maddalena, Morel, and, Solimini in [1] and Bernot, Caselles, and Morel in [4] proposed a Lagrangian formulation based on *irrigation patterns* χ that describe the position of each mass particle p at time t by $\chi(p, t)$. The difference between both articles is that in the latter the particle trajectories cannot be reparameterised without changing the transport cost. The viewpoint introduced by Xia in [2] is instead Eulerian, where only the flux of particles is described, discarding its dependence on the time variable t. A very interesting aspect of branched transport is its regularity theory as studied in several articles, among them [5] and [6] for the geometric regularity, [7] for the regularity of the tangents to the branched structure, [8] and [9] for the regularity of the landscape function, [10] for the fractal regularity of the minimisers. Equivalence of the different models and formulations are instead the topic of [11,12]. Branched transport can also be modelled with curves in the Wasserstein space as in [13–15].