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A sharp quantitative version of Hales' isoperimetric honeycomb theorem

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ABSTRACT

We prove a sharp quantitative version of Hales' isoperimetric honeycomb theorem by exploiting a quantitative isoperimetric inequality for polygons and an improved convergence theorem for planar bubble clusters. Further applications include the description of isoperimetric tilings of the torus with respect to almost unit-area constraints or with respect to almost flat Riemannian metrics.

RÉSUMÉ

On démontre une version quantitative optimale du théorème d'isopérimétrie de Hales en exploitant une inégalité isopérimétrique quantitative sur les polygones et une convergence améliorée pour les amas planaires de bulles. Des conséquences incluent la description de pavages isopérimétriques du tore par des contraintes presque unitaires ou par des métriques riemanniennes presque plates.

1. Introduction

The isoperimetric nature of the planar "honeycomb tiling" has been apparent since antiquity. Referring to [12, Section 15.1] for a brief historical account on this problem, we just recall here that Hales' isoperimetric theorem, see inequality (1.2) below, gives a precise formulation of this intuitive idea. Our goal here is to strengthen Hales' theorem into a quantitative statement, similarly to what has been done with other isoperimetric theorems in recent years (see, for example, [6,7]).

Following [11, Chapters 29–30], we work in the framework of sets of finite perimeter. A N-*tiling* $\mathcal{E}$ of a two-dimensional torus $\mathcal{T}$ is a family $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ of sets of finite perimeter in $\mathcal{T}$ such that $|\mathcal{T} \setminus \bigcup_{h=1}^N \mathcal{E}(h)| = 0$

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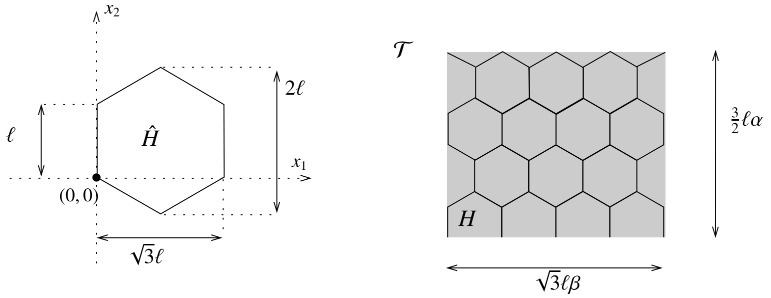

Fig. 1. Throughout the paper $\hat{H}$ denotes the unit-area regular hexagon in $\mathbb{R}^2$ depicted on the left and we set $H = \hat{H}/_{\approx}$. Since $|H| = 1$, one has $P(H) = 2(12)^{1/4}$, and the side-length of H is thus $\ell = (12)^{1/4}/3$. On the right, the torus $\mathcal{T}$ (depicted in gray) and the reference unit-area tiling $\mathcal{H}$ of $\mathcal{T}$ (with $\alpha = \beta = 4$). Notice that $N = |\mathcal{T}| = \alpha\,\beta$. The chambers of $\mathcal{H}$ are enumerated so that $\mathcal{H}(1) = H$, $\{\mathcal{H}(h)\}_{h=1}^{\beta}$ is the bottom row of hexagons in $\mathcal{T}$, and, more generally, if $0 \le k \le \alpha - 1$, then $\{\mathcal{H}(h)\}_{h=1+k\beta}^{(k+1)\beta}$ is the $(k+1)$th row of hexagons in $\mathcal{T}$.

and $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$ for every $h, k \in \mathbb{N}$, $h \ne k$. The volume of $\mathcal{E}$ is $\mathrm{vol}\,(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|)$, and the relative perimeter of $\mathcal{E}$ in $A \subset \mathcal{T}$ is given by

$$P(\mathcal{E}; A) = \frac{1}{2}\sum_{h=1}^{N} P(\mathcal{E}(h); A)\,,$$

(where $P(E; A) = \mathcal{H}^1(A \cap \partial E)$ if E is an open set with Lipschitz boundary), while the distance between two tilings $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$\mathrm{d}(\mathcal{E}, \mathcal{F}) = \frac{1}{2}\sum_{h=1}^{N} |\mathcal{E}(h)\Delta\mathcal{F}(h)|\,.$$

We say that $\mathcal{E}$ is a *unit-area tiling* of $\mathcal{T}$ if $|\mathcal{E}(h)| = 1$ for every $h = 1, \dots, N$. (In particular, in that case, it must be $N = |\mathcal{T}|$.) Let $\hat{H}$ denote the reference unit-area hexagon in $\mathbb{R}^2$ depicted in Fig. 1, so that $\ell = (12)^{1/4}/3$ is the side-length of $\hat{H}$. Given $\alpha, \beta \in \mathbb{N}$, let us consider the torus $\mathcal{T} = \mathcal{T}_{\alpha,\beta} = \mathbb{R}^2/_{\approx}$, where

$$(x_1, x_2) \approx (y_1, y_2) \qquad \text{if and only if} \qquad \exists h, k \in \mathbb{N} \text{ s.t.} \quad \begin{cases} x_1 = y_1 + h\,\beta\,\sqrt{3}\,\ell\,, \\ x_2 = y_2 + k\,\alpha\,\frac{3}{2}\,\ell\,, \end{cases}$$

and set $H = \hat{H}/_{\approx} \subset \mathcal{T}$. In order to avoid degenerate situations, *we shall always assume that*

$$\alpha \text{ is even and } \beta \ge 2\,. \tag{1.1}$$

In this way, H is a regular unit-area hexagon (i.e., the vertexes of $\hat{H}$ belong to six different equivalence classes) and one obtains a reference unit-area tiling $\mathcal{H} = \{\mathcal{H}(h)\}_{h=1}^{N}$ of $\mathcal{T}$ consisting of α rows and β columns of regular hexagons by considering translations of H by $(h\,\sqrt{3}\ell, 3\ell\,k/2)$ $(h, k \in \mathbb{Z})$; see again Fig. 1. Under this assumption, *Hales' isoperimetric honeycomb theorem* asserts that

$$P(\mathcal{E}) \ge P(\mathcal{H})\,, \tag{1.2}$$

whenever $\mathcal{E}$ is a unit-area tiling of $\mathcal{T}$, and that $P(\mathcal{E}) = P(\mathcal{H})$ if and only if (up to a relabeling of the chambers of $\mathcal{E}$) one has $\mathcal{E}(h) = v + \mathcal{H}(h)$ for every $h = 1, \dots, N$ and for some $v = (t\sqrt{3}\ell, s\ell)$ with $s, t \in [0, 1]$. Our first main result strengthens this isoperimetric theorem in a sharp quantitative way.