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A gradient flow approach to large deviations for diffusion processes

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ABSTRACT

In this work, we investigate links between the formulation of the flow of marginals of reversible diffusion processes as gradient flows in the space of probability measures and path wise large deviation principles for sequences of such processes. An equivalence between the LDP principle and Gamma-convergence for a sequence of functionals appearing in the gradient flow formulation is proved. As an application, we study large deviations from the hydrodynamic limit for two variants of the Ginzburg–Landau model endowed with Kawasaki dynamics.

RÉSUMÉ

Dans cet article, on étudie les liens entre la description du flot de marginales de processus de diffusions réversibles comme flot-gradient dans un espace de mesures de probabilités et la théorie des grandes déviations pour des suites de trajectoires de tels processus. Une équivalence entre principe de grandes déviations et Gamma-convergence d'une suite de fonctionnelles apparaissant dans la formulation en flot-gradient est démontrée. Comme application, on étudie les grandes déviations par rapport à la limite hydrodynamique pour deux variantes du modèle de Ginzburg–Landau muni d'une dynamique de Kawasaki.

1. Introduction

In this work, we are interested in the links between the gradient flow formulation of the flow of marginals of stochastic differential equations, and path wise large deviations for sequences of such processes.

Interest in gradient flows on the space of probability measures goes back to [19], where it was observed that the heat equation can be viewed as the gradient flow of the entropy

$$\mathrm{Ent}(\rho) = \int \rho \log \rho dx$$

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for the Wasserstein distance W_2. Note that what we will call here entropy is the negative of the physical entropy. This was later developed into a notion of formal Riemannian structure on $\mathcal{P}(\mathbb{R}^n)$ by Otto in [31]. While a powerful tool to predict the behavior of certain partial differential equations, the point of view of Otto is formal, and we must rely on other tools for proofs.

Another point of view was developed by Ambrosio, Gigli and Savaré in [3], which uses the notion of 'minimizing movement' schemes, developed by De Giorgi and which first appeared in [7], to provide a rigorous framework to define gradient flows on spaces of probability measures. It is based on the idea that gradient flows on $\mathbb{R}^n$ of the form

$$\dot{x}(t) = -\nabla F(x(t))$$

are the only solutions of

$$F(x(T)) - F(x(0)) + \frac{1}{2}\int_0^T |\nabla F(x(t))|^2 dt + \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt = 0. \tag{1.1}$$

While the usual gradient flow equation only makes sense in a Riemannian setting (at least in the classical sense), this alternative formulation can be given a meaning in a purely metric setting, as long as we can define a 'length of the gradient' functional $|\nabla F|$. Section 3.1 concerns this formulation in the setting of the space of probability measures on $\mathbb{R}^n$ endowed with a Wasserstein distance W_2, when the functional F is the relative entropy with respect to a nonnegative measure μ, that is

$$\mathrm{Ent}_\mu(\nu) := \int f \log f d\mu$$

if $\nu = f\mu$, and $+\infty$ if ν is not absolutely continuous with respect to μ. This is the framework developed in the first sections of [3].

Several recent papers have been interested in using abstract gradient flow formulations to study convergence of sequences of solutions to partial differential equations. One method, tailored for the case of diffusion processes and based on the discrete approximation of gradient flows, has been devised in [4]. Another, more general, method has been presented in [36] (generalizing previous results of [35]). It consists in studying the asymptotic behavior of sequence of functionals of the form (1.1) for given functions F_n. Informally, it consists in showing that, if the sequence converges in a certain sense to a limiting functional F_∞, we can directly identify limits of solutions of (1.1) as gradients flows for the limiting function F_∞.

In the context of statistical physics, the method developed in [36] can be used to prove convergence to the hydrodynamic limit for some models of interacting diffusion processes, such as the Ginzburg–Landau model (see [18] or [17] for a presentation of the model, and its hydrodynamic limit). Such results consist in proving convergence in probability of the dynamics of some family of N-particle systems to a deterministic limiting object as the number of particles N goes to infinity. The limit generally appears as the solution to some partial differential equation. Gradient flows have also been used in [32] to study convergence to the hydrodynamic limit for microscopic models describing heat conduction.

Our aim here is to use the notion of gradient flows to study large deviations from the hydrodynamic limit for interacting spin systems. Such a result consists in proving that the probabilities of a significant deviation from the hydrodynamic limit decays exponentially fast in the system size. A standard textbook on the topic of large deviations is [10], and [21] contains a review of the literature in the context of large deviations from the hydrodynamic limit for particle systems.

In the work [1] (and its sequels [2] and [5]), links between gradient flows in spaces of probability measures and large deviations have been investigated for many examples of processes arising in statistical physics.