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Special birational structures on non-Kählerian complex surfaces

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A B S T R A C T

We investigate the following conjecture: all compact non-Kählerian complex surfaces admit birational structures. After Inoue–Kobayashi–Ochiai, the remaining cases to study are essentially surfaces in class VII_0^+. We show that Kato surfaces with a cycle and only one branch of rational curves admit a special birational structure given by new normal forms of contracting germs in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$. In particular all surfaces S with GSS and second Betti number satisfying $0 \leq b_2(S) \leq 3$ admit a birational structure. From the existence of a special birational structure we deduce a developing meromorphic mappings $\tilde{S} \to \mathbb{P}^2(\mathbb{C})$ from the universal cover of S to $\mathbb{P}^2(\mathbb{C})$ which blows down an infinite number of rational curves. From this mapping we recover a GSS.

R É S U M É

On étudie la conjecture suivante : toute surface complexe compacte non-kählerienne admet une structure birationnelle. D'après Inoue–Kobayashi–Ochiai, les cas restants à étudier sont essentiellement les surfaces de la classe VII_0^+. On démontre que les surfaces de Kato qui ont un cycle avec un seul arbre de courbes rationnelles admettent une structure birationnelle spéciale définie par de nouvelles formes normales de germes de contractions dans le groupe de Cremona $Bir(\mathbb{P}^2(\mathbb{C}))$. En particulier toute surface S contenant une coquille sphérique globale (CSG) et pour laquelle le second nombre de Betti vérifie $0 \leq b_2(S) \leq 3$ admet une structure birationnelle. De l'existence d'une structure birationnelle on déduit une application méromorphe développante $\tilde{S} \to \mathbb{P}^2(\mathbb{C})$ du revêtement universel de S dans $\mathbb{P}^2(\mathbb{C})$ qui écrase une infinité de courbes rationnelles. Cette application permet de reconstituer la CSG.

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1. Introduction

What is the best G-structure on a compact manifold? The classification of Inoue–Kobayashi–Ochiai [14,18] shows that all compact complex non-Kählerian surfaces but some Hopf surfaces and surfaces in class VII_0^+ (i.e. in class VII_0 with $b_2 > 0$) admit affine structures. In view of the explicit construction of Kato surfaces (i.e. minimal surfaces S containing a global spherical shell (GSS) with $b_2(S) > 0$) and the particular cases of Enoki surfaces and Inoue–Hirzebruch surfaces the best G-structure should be obtained for a subgroup of $Bir(\mathbb{P}^2(\mathbb{C}))$. It justifies the following

Conjecture: All compact complex non-Kähler surfaces admit birational structures.

The conjecture is clearly satisfied for all Hopf surfaces because they are defined by an invertible contracting polynomial mapping. Remains the case of surfaces in class VII_0^+. Since the only known surfaces in class VII_0^+ are Kato surfaces and since it is conjectured that there are no others, this article focuses on the following problem: Do compact surfaces with GSS admit a birational structure, i.e. is there an atlas with transition mappings in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$. As stronger requirement, is there in each conjugation class of contracting germs of the form $\Pi\sigma$ (or of strict germs, following Favre terminology [12]) a birational representative? Clearly $\Pi\sigma$ is birational if and only if σ is birational.

Known results:

- If S is a Enoki surface (see [7]) or a Inoue–Hirzebruch surface (see [4]) with second Betti number $b_2(S) = n$, known normal forms, namely

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2), \quad 0 < |t| < 1,$$

  and

$$N(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s),$$

  respectively, are birational. Here $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl(2, \mathbb{Z})$ is the composition of n matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

  with at least one of the second type.
- If S is of intermediate type (see definition in section 2), there are normal forms due to C. Favre [12]

$$F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2), z_2^k), \qquad \lambda \in \mathbb{C}^\star,\ \mathfrak{s} \in \mathbb{N}^\star,\ k \geq 2,$$

  where P is a special polynomial. These normal forms are adapted to logarithmic deformations and show the existence of a foliation, however *are not birational*. In [21] K. Oeljeklaus and M. Toma explain how to recover second Betti number n which is now hidden and give coarse moduli spaces of surfaces with fixed intersection matrix.
- Some special cases of intermediate surfaces are obtained from Hénon mappings H or composition of Hénon mappings. More precisely, the germ of H at the fixed point at infinity is strict, hence gives a surface with a GSS [13,9]. These germs are birational.