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An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients

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ABSTRACT

In this paper we establish the $C^{1,\beta}$ regularity of the regular part of the free boundary in the Signorini problem for elliptic operators with variable Lipschitz coefficients. This work is a continuation of the recent paper [12], where two of us established the interior optimal regularity of the solution. Two of the central results of the present work are a new monotonicity formula and a new epiperimetric inequality.

RÉSUMÉ

Dans cet article on établit la régularité $C^{1,\beta}$ de la partie régulière de la frontière libre dans le problème de Signorini pour des opérateurs elliptiques à coefficients variables lipchitziens. Ce travail est une continuation de l'étude récente [12], dans laquelle deux d'entre-nous ont établi la régularité intérieur optimale de la solution. Une nouvelle formule de monotonie et une nouvelle inégalité epipérimétrique présentent deux résultats fondamentaux de cet article.

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1. Introduction

1.1. Statement of the problem and main assumptions

The purpose of the present paper is to establish the $C^{1,\beta}$ regularity of the free boundary near so-called regular points in the Signorini problem for elliptic operators with variable Lipschitz coefficients. Although this work represents a continuation of the recent paper [12], where two of us established the interior optimal regularity of the solution, proving the regularity of the free boundary has posed some major new challenges. Two of the central results of the present work are a new monotonicity formula (Theorem 4.3) and a new epiperimetric inequality (Theorem 6.3). Both of these results have been inspired by those originally obtained by Weiss in [21] for the classical obstacle problem, but the adaptation to the Signorini problem has required a substantial amount of new ideas.

The lower-dimensional (or *thin*) *obstacle problem* consists of minimizing the (generalized) Dirichlet energy

$$\min_{u\in\mathcal{K}}\int_{\Omega}\langle A(x)\nabla u,\nabla u\rangle dx, \tag{1.1}$$

where u ranges in the closed convex set

$$\mathcal{K}=\mathcal{K}_{g,\varphi}=\{u\in W^{1,2}(\Omega)\mid u=g \text{ on } \partial\Omega,\ u\geq\varphi \text{ on } \mathcal{M}\cap\Omega\}.$$

Here, $\Omega\subset\mathbb{R}^n$ is a given bounded open set, $\mathcal{M}$ is a codimension one manifold which separates Ω into two parts, g is a boundary datum and the function $\varphi:\mathcal{M}\to\mathbb{R}$ represents the lower-dimensional, or thin, obstacle. The functions g and φ are required to satisfy the standard compatibility condition $g\geq\varphi$ on $\partial\Omega\cap\mathcal{M}$. This problem is known also as (scalar) *Signorini problem*, as the minimizers satisfy Signorini conditions on $\mathcal{M}$ (see (1.7)–(1.9) below in the case of flat $\mathcal{M}$).

Our assumptions on the matrix-valued function $x\mapsto A(x)=[a_{ij}(x)]$ in (1.1) are that $A(x)$ is symmetric, uniformly elliptic, and Lipschitz continuous (in short $A\in C^{0,1}$). Namely:

$$a_{ij}(x)=a_{ji}(x)\quad \text{for } i,j=1,\dots,n, \text{ and every } x\in\Omega; \tag{1.2}$$

there exists $\lambda>0$ such that for every $x\in\Omega$ and $\xi\in\mathbb{R}^n$, one has

$$\lambda|\xi|^2\leq\langle A(x)\xi,\xi\rangle\leq\lambda^{-1}|\xi|^2; \tag{1.3}$$

there exists $Q\geq 0$ such that

$$|a_{ij}(x)-a_{ij}(y)|\leq Q|x-y|,\quad x,y\in\Omega. \tag{1.4}$$

By standard methods in the calculus of variations it is known that, under appropriate assumptions on the data, the minimization problem (1.1) admits a unique solution $u\in\mathcal{K}$, see e.g. [10], or also [19]. The set

$$\Lambda^{\varphi}(u)=\{x\in\mathcal{M}\cap\Omega\mid u(x)=\varphi(x)\}$$

is known as the *coincidence set*, and its boundary (in the relative topology of $\mathcal{M}$)

$$\Gamma^{\varphi}(u)=\partial_{\mathcal{M}}\Lambda^{\varphi}(u)$$

is known as the *free boundary*. In this paper we are interested in the local regularity properties of $\Gamma^{\varphi}(u)$. When $\varphi=0$, we will write $\Lambda(u)$ and $\Gamma(u)$, instead of $\Lambda^0(u)$ and $\Gamma^0(u)$.