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Minimal solutions of a semilinear elliptic equation with a dynamical boundary condition

Marek Fila [a,*], Kazuhiro Ishige [b], Tatsuki Kawakami [c]

[a] *Department of Applied Mathematics and Statistics, Comenius University, 84248 Bratislava, Slovakia*
[b] *Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan*
[c] *Department of Mathematical Sciences, Osaka Prefecture University, Sakai 599-8531, Japan*

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ABSTRACT

We study properties of positive solutions of a semilinear elliptic equation with a linear dynamical boundary condition. We establish the semigroup property for minimal solutions, show that every local-in-time solution can be extended globally, and reveal a relationship between minimal solutions of the time-dependent problem and minimal solutions of a corresponding stationary problem.

RÉSUMÉ

On étudie les solutions positives d'une équation elliptique semi-linéaire avec une condition aux limites dynamique linéaire. On établit la propriété de semigroupe pour les solutions minimales, on montre que toute solution locale en temps se prolonge en une solution globale, et on met en lumière une relation entre les solutions minimales du problème d'évolution et les solutions minimales d'un problème stationnaire.

1. Introduction

This paper is concerned with minimal solutions of a semilinear elliptic equation with a dynamical boundary condition,

$$\begin{cases} -\Delta u = f(u), & x \in \mathbb{R}^N_+,\ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+,\ t > 0, \\ u(x,0) = \varphi(x'), & x = (x',0) \in \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: fila@fmph.uniba.sk (M. Fila), ishige@math.tohoku.ac.jp (K. Ishige), kawakami@ms.osakafu-u.ac.jp (T. Kawakami).

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where $N \geq 2$, $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N : x_N > 0\}$, Δ is the N-dimensional Laplacian (in x), $\partial_t := \partial/\partial t$, $\partial_\nu := -\partial/\partial x_N$, φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$ and

$$f \text{ is a nondecreasing continuous function in } \mathbb{R} \text{ such that } f(0) \geq 0. \tag{1.2}$$

In [12–14] the authors of this paper studied the existence/nonexistence and the large time behavior of the solutions of the problem

$$\begin{cases} -\Delta u = u^p, & x \in \mathbb{R}^N_+,\ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+,\ t > 0, \\ u(x,0) = \varphi(x') \geq 0, & x = (x', 0) \in \partial\mathbb{R}^N_+, \end{cases} \tag{1.3}$$

where $p > 1$, and proved the following.

(i) If problem (1.3) has a solution in $\mathbb{R}^N_+ \times (0,T]$ for some $T > 0$, then there exists a unique minimal solution of (1.3) in $\mathbb{R}^N_+ \times (0,T]$.

(ii) If

$$1 < p < p_* := \frac{N+1}{N-1},$$

then problem (1.3) possesses no nontrivial local-in-time solutions. If $p = p_*$, then problem (1.3) possesses no nontrivial global-in-time solutions.

(iii) Let $p \geq p_*$ and let $\varphi(x') = \kappa(1 + |x'|)^{-2/(p-1)}$ with $\kappa > 0$. If κ is sufficiently large, then problem (1.3) possesses no local-in-time solutions. On the other hand, if κ is sufficiently small, then a solution of (1.1) exists globally in time.

In this paper we study qualitative properties of the minimal solution of (1.1) and we show:

(1) The existence of local-in-time solutions implies the existence of global-in-time solutions. In particular, in the case $p = p_*$, there are no nontrivial local-in-time solutions. Furthermore, if u is a solution in $\mathbb{R}^N_+ \times (0,T]$ for some $T > 0$, then there exists a global-in-time solution U such that $U(x,t) = u(x,t)$ in $\mathbb{R}^N_+ \times (0,T]$.

(2) The minimal solution has the semigroup property.

Furthermore, we consider the problem

$$\begin{cases} -\Delta v = f(v) & \text{in} \quad \mathbb{R}^N_+, \\ v = \varphi & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{1.4}$$

where φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$, and reveal a relationship between the minimal solutions of (1.1) and (1.4).

The exponent p_* is known as the Brézis–Turner exponent which is critical for the boundedness of very weak nonnegative solutions of the problem

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$. See [6] and [26, Chapter I, Section 11].