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Nonlinear Korn inequalities in $\mathbb{R}^n$ and immersions in $W^{2,p}$, $p > n$, considered as functions of their metric tensors in $W^{1,p}$

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ABSTRACT

A nonlinear Korn inequality in $\mathbb{R}^n$ provides an upper bound of an appropriate distance between two smooth enough immersions defined over an open subset Ω of $\mathbb{R}^n$ in terms of the corresponding distance between their metric tensors.

Under the assumption that Ω only satisfies the uniform interior cone property, we first establish such a nonlinear Korn inequality for immersions in the space $\boldsymbol{W}^{2,p}(\Omega)$, $p > n$, hence with metric tensors in the space $\mathbb{W}^{1,p}(\Omega)$; our point of departure is a crucial comparison theorem between solutions in $\mathbb{W}^{1,p}(\Omega)$ of Pfaff systems, which is due to the second author.

Under the assumptions that Ω is simply-connected and has a Lipschitz-continuous boundary, we then show that such immersions in $\boldsymbol{W}^{2,p}(\Omega)$ can be considered as well-defined functions, up to an isometric equivalence relation $\boldsymbol{\mathcal{R}}(\Omega)$, of their metric tensors in $\mathbb{W}^{1,p}(\Omega)$ if the Riemann curvature tensors of these tensors vanish in $\mathcal{D}'(\Omega)$. We also show that the mapping defined in this fashion from the space $\mathbb{W}^{1,p}(\Omega)$ into the quotient set $\boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{R}}(\Omega)$ is locally Lipschitz-continuous.

Under the only assumption that Ω is simply-connected, we finally show that one can define an analogous mapping, this time acting from the space $\mathbb{W}^{1,p}_{\text{loc}}(\Omega)$, $p > n$, into the quotient set $\boldsymbol{W}^{2,p}_{\text{loc}}(\Omega)/\boldsymbol{\mathcal{R}}(\Omega)$, and that this mapping is continuous when these spaces are equipped with their natural Fréchet topologies.

RÉSUMÉ

Une inégalité de Korn non linéaire dans $\mathbb{R}^n$ fournit une majoration d'une distance appropriée entre deux immersions suffisament régulières définies sur un ouvert Ω de $\mathbb{R}^n$ en fonction de la distance correspondante entre leurs tenseurs métriques.

Sous l'hypothèse que l'ouvert Ω satisfait seulement la propriété du cône intérieur uniforme, on établit pour commencer une telle inégalité de Korn non linéaire pour des immersions dans l'espace $\boldsymbol{W}^{2,p}(\Omega)$, $p > n$, donc avec des tenseurs métriques dans l'espace $\mathbb{W}^{2,p}(\Omega)$; notre point de départ est un théorème crucial de comparaison entre solutions dans l'espace $\mathbb{W}^{1,p}(\Omega)$ de systèmes de Pfaff, qui est dû au second auteur.

Sous l'hypothèse que l'ouvert Ω est simplement connexe et de frontière lipschitzienne, on montre ensuite que de telles immersions dans $\boldsymbol{W}^{2,p}(\Omega)$ peuvent être considérées

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comme des fonctions bien définies, à une relation $\mathcal{R}(\Omega)$ d'équivalence isométrique près, de leurs tenseurs métriques dans $\mathbb{W}^{1,p}(\Omega)$ si les tenseurs de courbure de Riemann de ces derniers s'annulent dans $\mathcal{D}'(\Omega)$. On montre aussi que l'application aussi définie de l'espace $\mathbb{W}^{1,p}(\Omega)$ dans l'ensemble quotient $\boldsymbol{W}^{2,p}(\Omega)/\mathcal{R}(\Omega)$ est localement lipschitzienne.
Sous la seule hypothèse que l'ouvert Ω est simplement connexe, on montre enfin que l'on peut définir une application analogue, agissant cette fois de l'espace $\mathbb{W}^{1,p}_{\text{loc}}(\Omega)$, $p > n$, dans l'ensemble quotient $\boldsymbol{W}^{2,p}_{\text{loc}}(\Omega)/\mathcal{R}(\Omega)$, et que cette application est continue lorsque ces espaces sont munis de leur topologies naturelles de Fréchet.

1. Introduction

All the notations used, but not defined, in this introduction are defined in Section 2.

Throughout this paper, n designates an integer ≥ 2. Let Ω be an open subset of $\mathbb{R}^n$, let $\mathbb{E}^n$ denote the n-dimensional Euclidean space, and let a smooth enough immersion $\boldsymbol{\varphi} : \Omega \to \mathbb{E}^n$ be given (if $\boldsymbol{\varphi} \in \mathcal{C}^1(\Omega)$, this means that $\det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) \neq 0$ for all $x \in \Omega$; if $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p}(\Omega)$ for some $p \geq 1$, this means that $\det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) \neq 0$ for almost all $x \in \Omega$). The associated *metric tensor field* $\boldsymbol{C} : \Omega \to \mathbb{S}^n$ is then defined, possibly only almost-everywhere, by

$$\boldsymbol{C}(x) := \boldsymbol{\nabla}\boldsymbol{\varphi}(x)^T \boldsymbol{\nabla}\boldsymbol{\varphi}(x)$$

at the points $x \in \Omega$ where the matrix $\boldsymbol{\nabla}\boldsymbol{\varphi}(x)$ is defined; thus $\boldsymbol{C}(x)$ is a symmetric and positive-definite $n \times n$ matrix at those points. The adjective "metric" reflects that *"metric" notions* such as lengths of curves, areas, and volumes, inside the image $\boldsymbol{\varphi}(\Omega)$ of Ω under $\boldsymbol{\varphi}$, considered as being *isometrically imbedded in* $\mathbb{E}^n$, can be computed using the sole knowledge of the field $\boldsymbol{C} : \Omega \to \mathbb{S}^n$ (see, e.g., Section 8.2 in [3]).

Let $\mathbb{O}^n$ denote the set of all $n \times n$ orthogonal matrices. Two smooth enough immersions $\boldsymbol{\varphi} : \Omega \to \mathbb{E}^n$ and $\boldsymbol{\psi} : \Omega \to \mathbb{E}^n$ are said to be *isometrically equivalent* if there exist a vector $\boldsymbol{a} \in \mathbb{E}^n$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^n$ such that

$$\boldsymbol{\psi} = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\varphi}.$$

The resulting relation $\boldsymbol{\nabla}\boldsymbol{\psi}^T \boldsymbol{\nabla}\boldsymbol{\psi} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi}$ therefore shows that two isometrically equivalent immersions share the same metric tensor field.

The converse also holds under specific assumptions. For instance, if Ω is connected and $\boldsymbol{\varphi} \in \mathcal{C}^1(\Omega)$ and $\boldsymbol{\psi} \in \mathcal{C}^1(\Omega)$ satisfy $\boldsymbol{\nabla}\boldsymbol{\psi}^T \boldsymbol{\nabla}\boldsymbol{\psi} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi}$ in Ω, then $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are isometrically equivalent (see, e.g., Theorem 8.7-1 in [3]); if Ω is connected and $\boldsymbol{\varphi} \in \mathcal{C}^1(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ satisfy $\boldsymbol{\nabla}\boldsymbol{\psi}^T \boldsymbol{\nabla}\boldsymbol{\psi} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi}$ almost everywhere in Ω and the sign of $\det \boldsymbol{\nabla}\boldsymbol{\psi}$ is almost-everywhere the same as that of $\det \boldsymbol{\nabla}\boldsymbol{\varphi}$, then $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are likewise isometrically equivalent (see [6]).

A *nonlinear Korn inequality* is any inequality that provides an upper bound of some appropriate norm of the difference $\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}$ (only up to isometric equivalence; see the examples given below) between two smooth enough immersions $\boldsymbol{\varphi} : \Omega \to \mathbb{E}^n$ and $\tilde{\boldsymbol{\varphi}} : \Omega \to \mathbb{E}^n$ in terms of some appropriate norm of the difference $\boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi} - \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}$ between their associated metric tensors, or of the difference $(\boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi})^{1/2} - (\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}})^{1/2}$ between the square roots of the same tensors; "nonlinear" reflects here that the difference $\boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi} - \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}$, or the difference $(\boldsymbol{\nabla}\boldsymbol{\varphi}^T \boldsymbol{\nabla}\boldsymbol{\varphi})^{1/2} - (\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}})^{1/2}$, is not approximated by its "linear part" with respect to the difference $\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}$, as is the case in the classical *linear* Korn inequalities (some examples of which are given below).

Nonlinear Korn inequalities were first established in 1961 and 1972 by John [18,19], and by Kohn [21] in 1982. In these pioneering contributions, these inequalities were, however, restricted to the special case