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Stability results on the smoothness of optimal transport maps with general costs

Shibing Chen [a], Alessio Figalli [b,*]

[a] *Mathematical Science Institute, The Australian National University, Canberra ACT 2601, Australia*
[b] *Department of Mathematics, The University of Texas at Austin, 2515 Speedway Stop C1200, Austin TX 78712-1202, USA*

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ABSTRACT

We prove some stability results concerning the smoothness of optimal transport maps with general cost functions. In particular, we show that the smoothness of optimal transport maps is an open condition with respect to the cost function and the densities. As a consequence, we obtain regularity for a large class of transport problems, where the cost does not necessarily satisfy the MTW condition.

RÉSUMÉ

On montre des résultats de stabilité concernant la régularité du transport optimal avec des fonctions de coût générales. En particulier, on montre que la régularité du transport est une condition ouverte par rapport à la fonction de coût et aux densités. En conséquence, on obtient la régularité pour une large classe de problèmes de transport dont le coût ne satisfait pas nécessairement la condition MTW.

1. Introduction

Given a source domain $X \subset \mathbb{R}^n$ associated with density $f : X \to \mathbb{R}^+$, a target domain $Y \subset \mathbb{R}^n$ associated with density $g : Y \to \mathbb{R}^+$, and a cost function $c : X \times Y \to \mathbb{R}$, the optimal transport problem consists in finding, among all transport maps (i.e., all maps $T : X \to Y$ such that $T_\sharp f = g$), a transport map which minimizes the total transportation cost

$$\int_X c(x, T(x))\, f(x)\, dx.$$

* Corresponding author.
E-mail addresses: shibing.chen@anu.edu.au (S. Chen), figalli@math.utexas.edu (A. Figalli).

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It is by now well-known that, under some rather general assumptions on the cost c, there exists a unique transport map T (see Section 2 for more details). Then, a very natural and important question becomes the following:

If the data f, g, X, Y, c are smooth, is T smooth as well?

While this question is well understood when c is the squared distance function in $\mathbb{R}^n$ (see for instance [1,3,5,6,33,13,15], [14, Chapter 4.5]), the regularity of optimal transport maps with general cost has been for long time a fundamental open problem in the theory of optimal transportation. In 2005, Ma, Trudinger, and Wang [30] found the following fourth order condition (now called MTW condition after their names) on the cost function:

$$\sum_{i,j,k,l,p,q,r,s} c^{p,q}(c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}\xi_i\xi_j\eta_k\eta_l \geq 0 \qquad \text{in } X \times Y \tag{1.1}$$

for all ξ, $\eta \in \mathbb{R}^n$ satisfying $\xi \perp \eta$, where lower indices before (resp. after) the comma indicate derivatives with respect to x (resp. y) (so for instance $c_{i,j} = \frac{\partial^2 c}{\partial x_i \partial y_j}$), $(c^{i,j})$ is the inverse of $(c_{i,j})$, and all derivatives are evaluated at $(x, y) \in X \times Y$. Under the above condition, they proved in [30] that if the densities are positive and smooth and the domains satisfy some suitable convexity assumptions, then the optimal map is smooth (see also [31,32]). Later Loeper [27] showed that the MTW condition is actually *necessary* for the smoothness of optimal transport maps. More precisely, if the cost function does not satisfy the MTW condition, Loeper constructed two smooth positive densities, supported on smooth domains satisfying the “right” convexity assumptions, for which the optimal map was not even continuous. After these two important works, many experts have contributed to develop a complete regularity theory of optimal transport problem under the MTW condition, to cite a few see [18,25,31,32,19,28,29,26,20,24,22,21,12,16,17].

Unfortunately, several interesting costs do not satisfy the MTW condition, for instance $c(x, y) = \frac{1}{p}|x-y|^p$ does not satisfy MTW condition when $p \in (1, 2) \cup (2, \infty)$, and actually the class of costs satisfying the MTW condition is very restricted. Recently, De Philippis and Figalli [10] obtained a partial regularity result for optimal transport problem with general cost without assuming neither the MTW condition nor any convexity on the domains. They managed to show that optimal maps are always smooth outside a closed set of measure zero. In a related direction, Caffarelli, Gonzáles, and Nguyen [7] obtained an interior $C^{2,\alpha}$ regularity result of optimal transport problem when the densities are C^α and the cost function is of the form $c(x, y) = \frac{1}{p}|x-y|^p$ with $2 < p < 2 + \epsilon$ for some $\epsilon \ll 1$. This interior regularity result was later extended by us to a global one, and generalized to a larger class of cost functions [8].

Motivated by the recent results and techniques developed in [10,8], in this paper we show the following stability statement: consider the optimal transport problem from (X, f) to (Y, g) with cost c, and suppose that the optimal maps T_u and $T^*_{u^c}$ sending respectively f to g and g to f have some (suitable) degree of smoothness. Then, if we perturb the problem sightly, regularity persists. More precisely, consider the optimal transport problem from $(X, \tilde{f})$ to $(Y, \tilde{g})$ with cost $\tilde{c}$, and assume that $\tilde{f}$ and $\tilde{g}$ are close to f and g in C^0 norm respectively, and $\tilde{c}$ is close to c in C^2 norm. Then the corresponding optimal transport maps $T_{\tilde{u}}$ and $T^*_{\tilde{u}^c}$ enjoy the same smoothness as T_u and $T^*_{u^c}$. This is particularly interesting since, even if c satisfies the MTW condition, $\tilde{c}$ may not satisfy it.

The paper is organized as follows. In section 2 we introduce some notation and state our main results. Then, in Section 3, we collect all the new ingredients that we need to apply the arguments in [10,8], and finally in the last section we prove our main results.

2. Preliminaries and main results

We begin by introducing some conditions that should be satisfied by the cost. Here and in the following, X and Y are two bounded open subsets of $\mathbb{R}^n$.