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Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients

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ABSTRACT

In this paper, we investigate the limiting absorption principle associated to and the well-posedness of the Helmholtz equations with sign changing coefficients which are used to model negative index materials. Using the reflecting technique introduced in [26], we first derive Cauchy problems from these equations. The limiting absorption principle and the well-posedness are then obtained via various a priori estimates for these Cauchy problems. Three approaches are proposed to obtain the a priori estimates. The first one follows from a priori estimates of elliptic systems equipped with complementing boundary conditions due to Agmon, Douglis, and Nirenberg in their classic work [1]. The second approach, which complements the first one, is variational and based on the Dirichlet principle. The last approach, which complements the second one, is also variational and uses the multiplier technique. Using these approaches, we are able to obtain new results on the well-posedness of these equations for which the conditions on the coefficients are imposed "partially" or "not strictly" on the interfaces of sign changing coefficients. This allows us to rediscover and extend known results obtained by the integral method, the pseudo differential operator theory, and the T-coercivity approach. The unique solution, obtained by the limiting absorption principle, is **not** in $H^1_{loc}(\mathbb{R}^d)$ as usual and possibly **not even** in $L^2_{loc}(\mathbb{R}^d)$. The optimality of our results is also discussed.

RÉSUMÉ

Dans cet article, on étudie le principe d'absorption limite et le caractère bien posé des équations de Helmholtz avec changements de signe des coefficients, ce qui modélise des matériaux d'indice négatif. En utilisant la technique de réflexion introduite dans [26], on dérive d'abord des problèmes de Cauchy. Le principe d'absorption limite et le caractère bien posé sont ensuite obtenus grâce à des estimations a priori pour ces problèmes. Trois approches sont proposées pour obtenir ces estimations. La première utilise les estimations a priori des systèmes elliptiques pour des conditions aux limites complémentaires dans l'ouvrage classique [1] d'Agmon, Douglis et Nirenberg. La deuxième approche, qui complète la première, est variationnelle et utilise le principe de Dirichlet. La dernière approche, qui complète la seconde, est également variationnelle et utilise la technique du multiplicateur. Utilisant ces approches, on peut obtenir des nouveaux résultats sur le caractère bien posé de ces équations, pour lesquelles les conditions sur les coefficients sont imposées "partiellement" ou "pas strictement" sur les interfaces où les coefficients

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changent la signe. Cela permet de redécouvrir et d'étendre les résultats connus obtenus par la méthode intégrale, la théorie des opérateurs pseudo differentiels, et l'approche T-coercivité. La solution unique, obtenue par le principe d'absorption limite, **n'est pas** dans $H^1_{loc}(\mathbb{R}^d)$ comme d'habitude et **n'est peut-être même pas** dans $L^2_{loc}(\mathbb{R}^d)$. L'optimalité de nos résultats est également discutée.

1. Introduction

This paper deals with the Helmholtz equation with sign changing coefficients which are used to model negative index materials (NIMs). NIMs were first investigated theoretically by Veselago in [44]. The existence of such materials was confirmed by Shelby, Smith, and Schultz in [42]. The study of NIMs has attracted a lot of attention in the scientific community thanks to their many possible applications such as superlensing and cloaking using complementary media, and cloaking a source via anomalous localized resonance.

We next mention briefly these three applications of NIMs. Superlensing using NIMs was suggested by Veselago in [44] for a slab lens (a slab of index -1) using the ray theory. Later, cylindrical lenses in the two dimensional quasistatic regime, the Veselago slab lens and cylindrical lenses in the finite frequency regime, and spherical lenses in the finite frequency regime were studied by Nicorovici, McPhedran, and Milton in [36], Pendry in [38,39], and Pendry and Ramakrishna in [41] respectively for constant isotropic objects. Superlensing using NIMs (or more precisely using complementary media) for arbitrary objects in the acoustic and electromagnetic settings was established in [27,31] for schemes inspired by [36,39,41] and guided by the concept of reflecting complementary media introduced and studied in [26]. Cloaking using complementary media was suggested and investigated numerically by Lai et al. in [18]. Cloaking an arbitrary inhomogeneous object using complementary media was proved in [30] for the quasi-static regime and later extended in [35] for the finite frequency regime. The schemes used there are inspired by [18] and [26]. Cloaking a source via anomalous localized resonance was discovered by Milton and Nicorovici for constant symmetric plasmonic structures in the two dimensional quasistatic regime in [22] (see also [24,36]) for dipoles. Cloaking an arbitrary source concentrated on a manifold of codimension 1 in an arbitrary medium via anomalous localized resonance was proposed and established in [28,29,33]. Other contributions are [3,4,11,17,34] in which special structures and partial aspects were investigated. A survey on the mathematics progress of these applications can be found in [32]. It is worthy noting that in the applications of NIMs mentioned above, the localized resonance, i.e., the field blows up in some regions and remains bounded in some others as the loss goes to 0, might appear.

In this paper, we investigate the well-posedness of the Helmholtz equation with sign changing coefficients: the stability aspect. To ensure to obtain physical solutions, we also study the limiting absorption principle associated to this equation. Let $k > 0$ and let A be a (real) uniformly elliptic symmetric matrix defined in $\mathbb{R}^d$ ($d \ge 2$), and Σ be a bounded real function defined in $\mathbb{R}^d$ (hence Σ can take both positive and negative values). Assume that

$$A(x) = I \text{ in } \mathbb{R}^d \setminus B_{R_0}, \quad A \text{ is piecewise } C^1,\text{[1]}$$

and

$$\Sigma(x) = 1 \text{ in } \mathbb{R}^d \setminus B_{R_0},$$

for some $R_0 > 0$. Here and in what follows, for $R > 0$, B_R denotes the open ball in $\mathbb{R}^d$ centered at the origin and of radius R. Let $D \subset\subset B_{R_0}$ be a bounded open subset in $\mathbb{R}^d$ of class C^2. Set, for $\delta \ge 0$,

[1] The smoothness assumption of A is only used in the proof of the uniqueness where the unique continuation is applied.