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Observation from measurable sets for parabolic analytic evolutions and applications ☆

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ABSTRACT

We find new quantitative estimates on the space–time analyticity of solutions to linear parabolic evolutions with time-independent analytic coefficients and apply them to obtain observability inequalities for its solutions over measurable sets.

RÉSUMÉ

On obtient de nouvelles estimations quantitatives sur l'analyticité en espace et en temps des solutions d'équations d'évolution linéaires paraboliques à coefficients analytiques indépendants du temps, et on les utilise pour obtenir des inégalités d'observabilité pour leurs solutions sur des ensembles mesurables.

1. Introduction

Mixing up ideas developed in [34], [2] and [28], it was shown in [3] that the heat equation over bounded domains Ω in $\mathbb{R}^n$ can be null controlled at all times $T > 0$ with interior and bounded controls acting over space–time measurable sets $\mathcal{D} \subset \Omega \times (0, T)$ with positive Lebesgue measure, when Ω is a Lipschitz polyhedron or a C^1 domain in $\mathbb{R}^n$. Apraiz et al. [3] also established the boundary null-controllability with bounded controls over measurable sets $\mathcal{J} \subset \partial\Omega \times (0, T)$ with positive surface measure.

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In this work we explain the techniques necessary to apply the same methods in [3] in order to obtain the interior and boundary null controllability of some higher order or non self-adjoint parabolic evolutions with time-independent analytic coefficients over analytic domains Ω of $\mathbb{R}^n$ and with bounded controls acting over measurable sets with positive measure. We also show the null-controllability with controls acting over possibly different measurable regions over each component of the Dirichlet data of higher order parabolic equations or over each component of the solution to second order parabolic systems; both at the interior and at the boundary. Finally, we show that the same methods imply the null-controllability of some not completely uncoupled parabolic systems with bounded interior controls acting over only one of the components of the system and on measurable regions.

We explain the technical details for parabolic higher order equations with constant coefficients and for second order systems with time independent analytic coefficients. We believe that this set of examples makes clear to the experts that the combination of the methods in [34,2,28] with others here imply analog results to those in [3] for parabolic evolutions associated to possibly non self-adjoint higher order elliptic equations or second order systems with time independent analytic coefficients over analytic domains: existence of bounded null-controls acting over measurable sets and the uniqueness and bang–bang property of certain optimal controls.

Throughout the work T denotes a positive time. Without loss of generality for the results we prove, we may very well assume $0 < T \le 1$. $\Omega \subset \mathbb{R}^n$ is a bounded domain with analytic boundary $\partial\Omega$, ν is the exterior unit normal vector to the boundary of Ω, $d\sigma$ denotes surface measure on $\partial\Omega$, $B_R(x_0)$ stands for the ball centered at x_0 of radius R, $B_R = B_R(0)$. For measurable sets $\omega \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n \times (0,T)$, $|\omega|$ and $|\mathcal{D}|$ stand for the Lebesgue measures of the sets; for measurable sets $\Gamma \subset \partial\Omega$ and $\mathcal{J}$ in $\partial\Omega \times (0,T)$, $|\Gamma|$ and $|\mathcal{J}|$ denote respectively their surface measures in $\partial\Omega$ and $\partial\Omega \times \mathbb{R}$. $|\alpha| = \alpha_1 + \cdots + \alpha_\ell$, when $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is an ℓ-tuple in $\mathbb{N}^\ell$, $\ell \ge 1$.

To describe the analyticity of the boundary of Ω we assume that there is some $\delta > 0$ such that for each x_0 in $\partial\Omega$ there is – after a translation and rotation – a new coordinate system (where $x_0 = 0$) and a real analytic function $\varphi : B'_\delta \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ verifying

$$\begin{aligned} &\varphi(0') = 0,\ |\partial^\alpha_{x'}\varphi(x')| \le |\alpha|!\,\delta^{-|\alpha|-1},\ \text{when } x' \in B'_\delta,\ \alpha \in \mathbb{N}^{n-1}, \\ &B_\delta \cap \Omega = B_\delta \cap \{(x', x_n) : x' \in B'_\delta,\ x_n > \varphi(x')\}, \\ &B_\delta \cap \partial\Omega = B_\delta \cap \{(x', x_n) : x' \in B'_\delta,\ x_n = \varphi(x')\}. \end{aligned} \tag{1.1}$$

The existence of the bounded null-controls acting over the measurable sets for the set of examples follows by standard duality arguments [6] from the following list of observability inequalities.

Theorem 1. *Let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure and $m \ge 1$. Then, there is $N = N(\Omega, T, m, \mathcal{D}, \delta)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \le N \int_{\mathcal{D}} |u(x,t)|\,dxdt$$

holds for all solutions u to

$$\begin{cases} \partial_t u + (-1)^m \Delta^m u = 0, & \text{in } \Omega \times (0,T), \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.2}$$

with u_0 in $L^2(\Omega)$.