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Lower bounds for pseudodifferential operators with a radial symbol

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ABSTRACT

In this paper we establish explicit lower bounds for pseudodifferential operators with a radial symbol. The proofs use classical Weyl calculus techniques and some useful, if not celebrated, properties of the Laguerre polynomials.

RÉSUMÉ

Dans cet article, on établit une borne inférieure pour les opérateurs pseudo-différentiels avec un symbole radial. La démonstration utilise des techniques classiques du calcul de Weyl ainsi que des propriétés connues vérifiées par les polynômes de Laguerre.

1. Introduction

If a function F defined on $\mathbb{R}^{2d}$ is smooth and has bounded derivatives, the Weyl calculus associates with it a pseudodifferential operator $Op_h^{Weyl}(F)$ which is bounded on $L^2(\mathbb{R}^d)$ and satisfies, for all f and g in $\mathcal{S}(\mathbb{R}^d)$,

$$\langle Op_h^{Weyl}(F)f, g\rangle = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} F(Z)H_h(f, g, Z)dZ, \tag{1.1}$$

where $H_h(f, g, \cdot)$ is the Wigner function

$$H_h(f, g, Z) = \int_{\mathbb{R}^d} e^{-\frac{i}{h}t\cdot\zeta} f\left(z + \frac{t}{2}\right)\overline{g\left(z - \frac{t}{2}\right)}dt \quad Z = (z, \zeta) \in \mathbb{R}^{2d}. \tag{1.2}$$

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For this form of the definition, see [15], [10] or [3, Chapter II, Proposition 14].

The different variants of Gårding's inequality prove that, if $F \geq 0$, the operator $Op_h^{Weyl}(F)$ is roughly ≥ 0. More precisely, according to the classical Gårding's inequality (see [7] or [10]), the non-negativity of F implies the existence of a positive constant C, independent of h, such that, for all sufficiently small h and for all f in $\mathcal{S}(\mathbb{R}^d)$:

$$\langle Op_h^{Weyl}(F)f, f\rangle \geq -Ch\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

See [11] for other similar results. This inequality holds for systems of operators, whereas the more precise Fefferman–Phong inequality [4] is valid only for scalar operators. Fefferman–Phong's inequality states that, under the same hypotheses as Gårding's inequality, one has, for all h in $(0,1)$ and all f in $\mathcal{S}(\mathbb{R}^d)$:

$$\langle Op_h^{Weyl}(F)f, f\rangle \geq -Ch^2\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

See [13] for these semiclassical versions. Sometimes the non-negativity of F implies the exact non-negativity of the operator, for example in the simple case when F depends on x or on ξ only. It is possible, too, to apply Melin's inequality [14]. To take only one example, let $F \geq 0$ attain its minimum only once, for a nondegenerate critical point. In this case (and in other analogous situations), Melin's inequality ensures the exact non-negativity of $Op_h^{Weyl}(F)$ for a sufficiently small h. See [2] or [9] for cases when the difference between $F(x,\xi)$ and its minimum is equivalent to a power, greater than 2, of the distance between (x,ξ) and the unique point where the minimum is attained.

In this article we are interested in the case when F is radial. We assume that there exists a function Φ defined on $\mathbb{R}$ such that

$$F(x,\xi) = \Phi\big(|x|^2 + |\xi|^2\big) \quad (x,\xi) \in \mathbb{R}^{2d}. \tag{1.5}$$

Moreover, we suppose that Φ is nondecreasing on $[0,\infty)$ and such that F is smooth, with bounded derivatives.

In this case, we aim at giving an explicit lower bound on the spectrum of the operator $Op_h^{Weyl}(F)$. The main result of this paper is the following theorem.

Theorem 1.1. *Let F be a smooth function defined on $\mathbb{R}^{2d}$, bounded as well as all its derivatives. Assume that F is of the form (1.5), where Φ is a non-decreasing function defined on $[0,\infty)$.*

Then for all f in $\mathcal{S}(\mathbb{R}^d)$,

$$\langle Op_h^{Weyl}(F)f, f\rangle \geq \frac{1}{h}\int_0^\infty \Phi(t)e^{-\frac{t}{h}}\,dt \quad \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.6}$$

Remarks. 1 – We do not need to assume that $\Phi \geq 0$ to ensure the non-negativity of the operator. The non-negativity of the integral suffices.

2 – In the case when Φ is not flat at the origin, assuming that Φ is infinitely differentiable at the origin (on the right side), let $m \geq 1$ be the smallest integer for which $\Phi^{(m)}(0) \neq 0$. Then one can see that

$$\frac{1}{h}\int_0^\infty \Phi(t)e^{-\frac{t}{h}}\,dt = \Phi(0) + \Phi^{(m)}(0)h^m + \mathcal{O}\big(h^{m+1}\big).$$

3 – The result can be applied to symbols F depending on the distance from another point (x_0,ξ_0) for, if $\tau F(x,\xi) = F(x+x_0, \xi+\xi_0)$ and $Tf(u) = e^{i(\xi_0/h)(u-x_0)}f(u-x_0)$, then

$$\langle Op_h^{Weyl}(\tau F)f, g\rangle = \langle Op_h^{Weyl}(F)Tf, Tg\rangle.$$