J. Math. Pures Appl. 103 (2015) 1276–1293

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators

Henri Berestycki [a], Italo Capuzzo Dolcetta [b,*], Alessio Porretta [c], Luca Rossi [d]

[a] *Ecole des Hautes Etudes en Sciences Sociales, CAMS, 190-198, av. de France, 75244 Paris, France*
[b] *Dipartimento di Matematica, Sapienza Università di Roma, piazzale A. Moro 3, 00185 Roma, Italy*
[c] *Dipartimento di Matematica, Università di Roma Tor Vergata, via della Ricerca Scientifica 1, 00133 Roma, Italy*
[d] *Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy*

ARTICLE INFO

Article history:
Received 17 October 2013
Available online 4 November 2014

To Louis Nirenberg with our affection and admiration

MSC:
35P30
35B50
35J70

Keywords:
Principal eigenvalue
Maximum Principle
Degenerate elliptic equation
Viscosity solution

ABSTRACT

We characterize the validity of the Maximum Principle in bounded domains for fully nonlinear degenerate elliptic operators in terms of the sign of a suitably defined generalized principal eigenvalue. Here, the maximum principle refers to the property of non-positivity of viscosity subsolutions of the Dirichlet problem. The new notion of generalized principal eigenvalue that we introduce here allows us to deal with arbitrary type of degeneracy of the elliptic operators. We further discuss the relations between this notion and other natural generalizations of the classical notion of principal eigenvalue, some of which have been previously introduced for particular classes of operators.

RÉSUMÉ

On caractérise la validité du principe du maximum pour des opérateurs elliptiques complètement non linéaires dégénérés au moyen du signe d'une valeur propre généralisée convenablement définie. Ici, le principe du maximum est entendu comme propriété des sous-solutions au sens viscosité du problème de Dirichlet d'être négatives ou nulles. La notion nouvelle de valeur propre principale introduite ici permet de traiter un cadre très général incluant les opérateurs elliptiques avec dégénérescence arbitraire. On examine les liens entre cette notion et d'autres extensions naturelles de la définition classique de valeur propre principale dont certaines ont été introduites précédemment pour des classes particulières d'opérateurs.

* Corresponding author.
E-mail address: capuzzo@mat.uniroma1.it (I. Capuzzo Dolcetta).

http://dx.doi.org/10.1016/j.matpur.2014.10.012
0021-7824/

1. Introduction

This paper is concerned with the Maximum Principle property for degenerate second order elliptic operators. Our aim is to characterize the validity of the Maximum Principle for arbitrary degeneracy of the operator – including the limiting cases of first and zero-order operators – in terms of the sign of a suitably defined generalized principal eigenvalue. Such a complete characterization is missing, as far as we know, even for the case of linear operators, which was of course our first motivation. Due to the possible loss of regularity, as well as of boundary conditions, which is caused by degeneracy of ellipticity, the appropriate framework to deal with this problem is, even in the linear case, that of viscosity solutions. This approach is of course not restricted to the linear case, so we study the question in the more general setting of homogeneous fully nonlinear degenerate elliptic operators $F(x, u, Du, D^2u)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ and $\mathcal{S}_N$ be the space of $n \times n$ symmetric matrices endowed with the usual partial order, with I being the identity matrix. A fully nonlinear operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is said to be *degenerate elliptic* if F is non-increasing in the matrix entry, see condition (H1) in the next section. The basic example to have in mind is that of linear operators in non-divergence form

$$F\big(x, u, Du, D^2u\big) = -\mathrm{Tr}\big(A(x)D^2u\big) - b(x) \cdot Du - c(x)u, \quad x \in \Omega,$$

where $A(x)$ is nonnegative definite.

We are interested in the following version of the Maximum Principle, **MP** in short:

Definition 1.1. The operator F satisfies **MP** in Ω if every viscosity subsolution $u \in USC(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} F\big(x, u, Du, D^2u\big) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

satisfies $u \le 0$ in $\overline{\Omega}$.

We denote by $USC(\overline{\Omega})$ the set of upper semicontinuous functions on $\overline{\Omega}$. It is worth pointing out that in the above definition both the PDE and the boundary conditions are understood in the viscosity sense (see Section 7 of [9]). Precisely, u is a subsolution of (1) if for all $\varphi \in C^2(\overline{\Omega})$ and $\xi \in \overline{\Omega}$, $\rho > 0$ such that $(u - \varphi)(\xi) = \max_{\overline{\Omega} \cap B_\rho(\xi)}(u - \varphi)$, it holds that

$$F\big(\xi, u(\xi), D\varphi(\xi), D^2\varphi(\xi)\big) \le 0 \quad \text{if } \xi \in \Omega,$$
$$\min\big[u(\xi), F\big(\xi, u(\xi), D\varphi(\xi), D^2\varphi(\xi)\big)\big] \le 0 \quad \text{if } \xi \in \partial\Omega.$$

Note, in particular, that the validity of the **MP** property implies that viscosity subsolutions cannot be positive on $\partial\Omega$, namely, the inequality $u \le 0$ on $\partial\Omega$ holds in the classical pointwise sense.

Before describing our results, let us recall some classical and more recent results concerning the Maximum Principle and the principal eigenvalue.

A standard result in the viscosity theory is that, under suitable continuity assumptions on the degenerate elliptic operator F, the Maximum Principle for viscosity subsolutions holds true if $r \mapsto F(x, r, p, X)$ is strictly increasing (see e.g. [9]). This is only a sufficient condition. It is well known that if Ω is a bounded smooth domain and F is a *uniformly elliptic* linear operator with smooth coefficients, then the validity of the Maximum Principle for classical subsolutions is equivalent to the positivity of the principal eigenvalue $\lambda_1(F, \Omega)$ associated with Dirichlet boundary condition. This eigenvalue is the bottom of the spectrum of the operator F acting on functions satisfying the Dirichlet boundary condition. It follows