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On a bilinear estimate in weak-Morrey spaces and uniqueness for Navier–Stokes equations

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ABSTRACT

This paper is concerned with the continuity of the bilinear term B associated with the mild formulation of the Navier–Stokes equations. We provide a new proof for the continuity of B in critical weak-Morrey spaces without using auxiliary norms of Besov type neither Kato time-weighted norms. As a byproduct, we reobtain the uniqueness of mild solutions in the class of continuous functions from $[0,T)$ to critical Morrey spaces. Our proof consists in estimates in block spaces (based on Lorentz spaces) that are preduals of Morrey–Lorentz spaces. For that, we need to obtain properties like interpolation of operators, duality, Hölder and Young type inequalities in such block spaces.

RÉSUMÉ

Cet article traite de la continuité du terme bilinéaire B associée à la formulation « mild » des équations de Navier–Stokes. On fournit une nouvelle démonstration de la continuité de B dans les espaces critiques de Morrey faibles sans utiliser des normes auxiliaires de type Besov ni des normes de Kato pondérées dans le temps. Comme sous-produit, on réobtient l'unicité de solutions « mild » dans la classe des fonctions continues de $[0,T)$ à valeurs dans les espaces critiques de Morrey. La démonstration consiste dans des estimations dans les espaces de blocs (basés sur les espaces de Lorentz) qui sont préduals d'espaces de Morrey–Lorentz. Pour cela on a besoin d'obtenir des propriétés telles que l'interpolation, la dualité et des inégalités de type Hölder et Young dans ces espaces de blocs.

1. Introduction

This paper is concerned with the free Navier–Stokes equations

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$$\frac{\partial u}{\partial t} - \Delta u + \mathbb{P}(u\nabla_x u) = 0, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \qquad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

where $n \geq 3$, $u = (u_j)_{j=1}^n$ is the velocity field and the Leray projector $\mathbb{P}$ is a matrix $n \times n$ with elements $(\mathbb{P})_{k,j} = \delta_{kj} + \mathcal{R}_k\mathcal{R}_j$, where $\mathcal{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ $(j = 1, 2, \ldots n)$ are the Riesz transforms.

According to Duhamel's principle, the Cauchy problem (1.1)–(1.3) is formally equivalent to the integral equation

$$u(t) = G(t)u_0 + B(u,u)(t), \tag{1.4}$$

where $G(t) = e^{\Delta t}$ denotes the heat semigroup, and B is the bilinear term

$$B(u,v)(t) = -\int_0^t \nabla_x G(t-s)\mathbb{P}(u \otimes v)(s)\, ds. \tag{1.5}$$

Divergence-free vector fields $u(x,t)$ satisfying (1.4) are called mild solutions for (1.1)–(1.3). The equations (1.1)–(1.2) present the scaling map

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t) \tag{1.6}$$

which induces the one (for the initial data)

$$u_0(x) \to \lambda u_0(\lambda x). \tag{1.7}$$

A Banach space X is called critical if it is invariant under (1.7), namely $\|u_0\|_X \approx \|\lambda u_0(\lambda x)\|_X$ for all $\lambda > 0$. Throughout the paper we assume that elements of functional spaces satisfy additionally the divergence-free condition in $\mathcal{S}'$, except for Sections 2 and 3. Also, spaces of scalar and vector functions are denoted abusively in the same way; for instance, we write $u \in L^p$ instead of the unpleasant $u \in (L^p)^n$.

There is a rich literature about existence of global-in-time mild solutions for (1.1)–(1.3) with small initial data in critical spaces such as homogeneous Sobolev $\dot{H}^{\frac{1}{2}}$ [10], L^n [18], Marcinkiewicz $L^{n,\infty}$ [1,38], PM^{n-1}-spaces [6,25], Besov $\dot{B}_{q,\infty}^{\frac{n}{q}-1}$ with $q > n$ [7], Morrey $\mathcal{M}_{q,n-q}$ [19,36] (see [9] for the non-critical case $\mathcal{M}_{q,\lambda}$ with $n - q < \lambda < n$), $\omega_0 = \nabla \times u_0 \in \mathcal{M}_{1,1}$ [13] (for the vorticity formulation in 3D), weak-Morrey $\mathcal{M}_{q,\infty,\lambda}$ with $\lambda = n - q$ [33], Fourier–Besov $F\dot{B}_{p,\infty}^{n-1-\frac{n}{p}}$ [17,21], Fourier–Herz $\mathcal{B}_r^{-1} = \mathcal{F}\mathcal{B}_{1,r}^{-1}$ with $r \in [1,2]$ [8,17,24], weak-Herz spaces [37], Besov–Morrey $\mathcal{N}_{q,\mu,\infty}^{\frac{n-\mu}{q}-1}$ [22] (see also [31]), and BMO^{-1} [23], among others (see [26] for a nice review). So far, BMO^{-1} and $\mathcal{N}_{1,\mu,\infty}^{n-\mu-1}$ (with $n - 1 < \mu < n$) are maximal critical spaces for (3DNS) in the sense that it is not known a critical space larger than them in which (NS) is globally solvable for small initial data.

The issue of uniqueness in these spaces is more subtle. Most of above existence results are proved by using Kato's approach (introduced in [10,18]) that consists in constructing a fixed point argument by using a suitable time-dependent space, whose norm is composed of two parts. The first is the norm of the persistence space $L^\infty((0,\infty); X)$ and the another is an auxiliary norm as the time-weighted norm $\sup_{t>0} t^\alpha \|u(\cdot,t)\|_Y$ where Y is a Banach space. In general, solutions are only time-weakly continuous at $t = 0^+$ because of the lack of strong continuity at $t = 0^+$ of the semigroup $\{e^{t\Delta}\}_{t\geq 0}$ in X. Considering the maximal closed subspace $\tilde{X}$ (endowed with $\|u\|_X$) in which $\{e^{t\Delta}\}_{t\geq 0}$ is continuous, one gets solutions in $C([0,T); \tilde{X})$ with large data and small $T > 0$. Another possibility is to consider $\tilde{X}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ in X. For most cases of X, two norms are necessary in order to control some integral terms when estimating