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Extrinsic Diophantine approximation on manifolds and fractals

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ABSTRACT

Fix $d \in \mathbb{N}$, and let $S \subseteq \mathbb{R}^d$ be either a real-analytic manifold or the limit set of an iterated function system (for example, S could be the Cantor set or the von Koch snowflake). An *extrinsic* Diophantine approximation to a point $\mathbf{x} \in S$ is a rational point $\mathbf{p}/q$ close to $\mathbf{x}$ which lies *outside* of S. These approximations correspond to a question asked by K. Mahler (1984) regarding the Cantor set. Our main result is an extrinsic analogue of Dirichlet's theorem. Specifically, we prove that if S does not contain a line segment, then for every $\mathbf{x} \in S \setminus \mathbb{Q}^d$, there exists $C > 0$ such that infinitely many vectors $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfy $\|\mathbf{x} - \mathbf{p}/q\| < C/q^{(d+1)/d}$. As this formula agrees with Dirichlet's theorem in $\mathbb{R}^d$ up to a multiplicative constant, one concludes that the set of rational approximants to points in S which lie outside of S is large. Furthermore, we deduce extrinsic analogues of the Jarník–Schmidt and Khinchin theorems from known results.

RÉSUMÉ

On fixe $d \in \mathbb{N}$, and $S \subseteq \mathbb{R}^d$ une variété analytique réelle ou bien un système de fonction itérée (par exemple S pourrait être l'ensemble de Cantor ou le flocon de neige de Von Koch). Une approximation diophantienne extrinsèque d'un point rationnel $\mathbf{p}/q$ voisin de $\mathbf{x} \in S$ est un point extérieur à S. Ces approximations répondent à une question posée par K. Mahler (1984) en considérant l'ensemble de Cantor. Le résultat fondamental de cet article est un analogue extrinsèque du théorème de Dirichlet. En particulier on démontre que si S ne contient pas de segment de droite alors pour tout point $\mathbf{x} \in S \setminus \mathbb{Q}^d$, on peut trouver une constante $C > 0$ tel qu'il existe une infinité de vecteurs $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ vérifiant $\|\mathbf{x} - \mathbf{p}/q\| < Cq^{(d+1)/d}$. Puisque cette formule est en accord avec le théorème de Dirichlet pour $\mathbb{R}^d$, à une constante multiplicative près, on conclut que l'ensemble des points rationnels approximants est grand. De plus on en déduit des analogues extrinsèques des théorèmes de Jarník–Schmidt et Kinchin à partir des résultats connus.

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1. Introduction

Fix $d \in \mathbb{N}$ and a set $S \subseteq \mathbb{R}^d$. One may divide the set of rational points into two disjoint classes: the class of rational points which lie on S, and the class of rational points which lie outside of S. Approximating points in S by rational points in S is called *intrinsic* approximation, while approximating points in S by rational points outside of S is called *extrinsic* approximation. More well-studied is the case where the approximations may come from either inside or outside S; in this case the approximations will be called *ambient*.

We shall be particularly interested in two classes of sets: S may be either the limit set of an iterated function system or a real-analytic manifold. Of particular prominence is the Cantor set,[1] of which K. Mahler [17] asked: "How close can irrational elements of Cantor's set be approximated by rational numbers (a) In Cantor's set, and (b) By rational numbers not in Cantor's set?" In our terminology, Mahler is asking about intrinsic and extrinsic approximation on the Cantor set, respectively. For both the limit sets of iterated function systems and for manifolds, there is already literature on both intrinsic and ambient approximations; see for example [1,4,11,12] and the references therein. By contrast, extrinsic approximation on algebraic varieties has been studied only briefly, in [8, Lemma 1], [9, Lemma 4.1.1], and [6, Lemma 1]. Each of these papers proved a lemma which stated that extrinsic rational approximations to points on algebraic varieties cannot be too close to the points they approximate.

In this paper, we analyze the theory of extrinsic approximation in more detail. Our main result (Theorem 1.1) is an extrinsic analogue of Dirichlet's theorem. We also describe results concerning extrinsic approximation which may be deduced from their intrinsic and ambient counterparts, namely analogues of the Jarník–Schmidt theorem and Khinchin's theorem.

Convention 1. The symbols $\lesssim$, $\gtrsim$, and $\asymp$ will denote multiplicative asymptotics. For example, $A \lesssim_K B$ means that there exists a constant $C > 0$ (the *implied constant*), depending only on K, such that $A \leq CB$. In general, dependence of the implied constant(s) on universal objects such as the set S will be omitted from the notation.

1.1. An extrinsic analogue of Dirichlet's theorem

Our main theorem is as follows:

Theorem 1.1. *Fix $d \in \mathbb{N}$, and let $S \subseteq \mathbb{R}^d$ be either*

(1) the limit set of an iterated function system[2] (cf. Definition 2.10), or
(2) a real-analytic manifold,

and suppose that S does not contain a line segment. Then for all $\mathbf{x} \in S \setminus \mathbb{Q}^d$, there exists $C = C_{\mathbf{x}} > 0$ such that infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfy

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \leq \frac{C}{q^{1+1/d}}. \tag{1.1}$$

Here and elsewhere $\|\cdot\|$ denotes the max norm. Moreover, the function $\mathbf{x} \mapsto C_{\mathbf{x}}$ is bounded on compact sets.

We recall that (the corollary of) Dirichlet's theorem in $\mathbb{R}^d$ states that for all $\mathbf{x} \in \mathbb{R}^d \setminus \mathbb{Q}^d$, there exist infinitely many $\mathbf{p}/q \in \mathbb{Q}^d$ satisfying (1.1) with $C = 1$. Thus Theorem 1.1 says that if S is as above, then for

URL: https://sites.google.com/site/davidsimmonsmath/ (D. Simmons).

[1] In this paper, the phrase "Cantor set" always refers to the ternary Cantor set.

[2] In this paper all iterated function systems are finite and consist of similarities.