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The dual Gromov–Hausdorff propinquity

Frédéric Latrémolière

Department of Mathematics, University of Denver, Denver, CO 80208, United States

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ABSTRACT

Motivated by the quest for an analogue of the Gromov–Hausdorff distance in noncommutative geometry which is well-behaved with respect to C*-algebraic structures, we propose a complete metric on the class of Leibniz quantum compact metric spaces, named the dual Gromov–Hausdorff propinquity. This metric resolves several important issues raised by recent research in noncommutative metric geometry: it makes *-isomorphism a necessary condition for distance zero, it is well-adapted to Leibniz seminorms, and — very importantly — is complete, unlike the quantum propinquity which we introduced earlier. Thus our new metric provides a natural tool for noncommutative metric geometry, designed to allow for the generalizations of techniques from metric geometry to C*-algebra theory.

RÉSUMÉ

Motivés par la quête d'une métrique analogue à la distance de Gromov–Hausdorff pour la géométrie noncommutative et adaptée aux C*-algèbres, on propose une distance complète sur la classe des espaces métriques compacts quantiques de Leibniz. Cette nouvelle distance, qu'on appelle la proximité duale de Gromov–Hausdorff, résout plusieurs problèmes importants que la recherche courante en géométrie métrique noncommutative a révélés. En particulier, il est nécessaire pour les C*-algèbres d'être isomorphes pour avoir une distance zéro, et tous les espaces quantiques compacts impliqués dans le calcul de la proximité duale sont de type Leibniz. En outre, notre distance est complète. Notre proximité duale de Gromov–Hausdorff est donc un nouvel outil naturel pour le développement de la géométrie métrique noncommutative.

1. Introduction

Noncommutative metric geometry proposes to study certain classes of noncommutative algebras as generalizations of algebras of Lipschitz functions over metric spaces. Inspired by Connes' pioneering work on

E-mail address: frederic@math.du.edu.
URL: http://www.math.du.edu/~frederic.

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noncommutative metric geometry [5,6] and, in particular, the construction of a metric on the state space of C*-algebras endowed with a spectral triple, Rieffel introduced in [33,34] the notion of a compact quantum metric space, and then defined the quantum Gromov–Hausdorff distance [43], a fascinating generalization of the Gromov–Hausdorff distance [10] to noncommutative geometry. Various examples of compact quantum metric spaces [35,31,27,25] and convergence results for the quantum Gromov–Hausdorff distance have since been established [43,25,19,36,27,38,40,39], often motivated by the desire to provide a formal framework to certain finite dimensional approximations of C*-algebras found in the mathematical physics literature (e.g. [7,30,44,29]). Furthermore, the introduction of noncommutative metric information on C*-algebras, encoded in special seminorms called Lip-norms, offers the possibility to extend techniques from metric geometry [11] to the realm of noncommutative geometry, opening a new avenue for the study of quantum spaces and their applications to mathematical physics.

To implement the extension of metric geometry to noncommutative geometry, we however require an analogue of the Gromov–Hausdorff distance which is well-behaved with respect to the underlying C*-algebraic structure, rather than with only the order structure on the self-adjoint part of C*-algebras, as with the quantum Gromov–Hausdorff distance. We propose such a metric in this paper, the dual Gromov–Hausdorff propinquity, which addresses several difficulties encountered during recent developments in noncommutative metric geometry [38,40,39,41,42], where the study of the behavior of such C*-algebraic structures as projective modules under metric convergence is undertaken. Indeed, our metric only involves Leibniz quantum compact metric spaces, i.e. quantum compact metric spaces described as C*-algebras endowed with Leibniz Lip-norms, and makes *-isomorphism of the underlying C*-algebras a necessary condition for distance zero. Moreover, our dual Gromov–Hausdorff propinquity is complete, which is an essential property of the Gromov–Hausdorff distance, and which differentiates our new metric from our earlier quantum Gromov–Hausdorff propinquity [22]. Our dual Gromov–Hausdorff propinquity dominates Rieffel's quantum Gromov–Hausdorff distance, and is dominated by the Gromov–Hausdorff distance when restricted to classical compact metric spaces. It thus offers a natural framework for a noncommutative theory of quantum metric spaces.

The model for a quantum compact metric space is derived from the following construction. Let (X, m) be a compact metric space. For any function $f : X \to \mathbb{C}$, we define the Lipschitz constant of f as:

$$\mathsf{Lip}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x, y)} : x, y \in X \text{ and } x \neq y \right\}, \tag{1}$$

which may be infinite. A function $f : X \to \mathbb{C}$ with a finite Lipschitz constant is called a Lipschitz function over X. The space of $\mathbb{C}$-valued Lipschitz functions is norm dense in the C*-algebra $C(X)$ of $\mathbb{C}$-valued continuous functions over X. Moreover, Lip is a seminorm on the space of $\mathbb{C}$-valued Lipschitz functions. However, we shall only work with the restriction of Lip to real-valued Lipschitz functions, which form a dense subset of the self-adjoint part $\mathfrak{sa}(C(X))$ of $C(X)$, because real-valued Lipschitz functions enjoy an extension property given by McShane's theorem [28] which will prove important in our work, as we shall see in a few paragraphs. Thus, unless explicitly stated otherwise, all the Lipschitz seminorms in this introduction are assumed to be restricted to real-valued Lipschitz functions.

A fundamental observation, due to Kantorovich, is that the dual seminorm of Lip induces a metric $\mathsf{mk}_{\mathsf{Lip}}$ on the state space $\mathscr{S}(C(X))$ of $C(X)$, i.e. the space of Radon probability measures on X, and the topology for this metric is given by the weak* topology restricted to $\mathscr{S}(C(X))$. Moreover, the restriction of $\mathsf{mk}_{\mathsf{Lip}}$ to the space X identified with the subset of Dirac probability measures in $\mathscr{S}(C(X))$ is given by m: thus the Lipschitz seminorm encodes all the metric information given by the metric m at the level of the C*-algebra $C(X)$.

The distance $\mathsf{mk}_{\mathsf{Lip}}$ is known as the Monge–Kantorovich metric, and was introduced by Kantorovich in [14] as part of his research on the transportation problem introduced by Monge in 1781. The original formulation of the distance $\mathsf{mk}_{\mathsf{Lip}}$ between two probability measures μ, ν on X involved minimizing $\int_{X\times X} \mathsf{m}\, d\gamma$ over all