J. Math. Pures Appl. 103 (2015) 557–574

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Topological sensitivity analysis for the modified Helmholtz equation under an impedance condition on the boundary of a hole

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ARTICLE INFO

Article history:
Received 7 April 2014
Available online 4 July 2014

MSC:
35J05
35J25
35Q93
34E05
34E15

Keywords:
Shape optimization
Topological sensitivity
Modified Helmholtz equation
Impedance condition
Inverse scattering
Cavity identification

ABSTRACT

The topological sensitivity analysis consists in providing an asymptotic expansion of a shape functional with respect to emerging of small holes in the interior of the domain occupied by the body. In this paper, such an expansion is obtained for the modified Helmholtz equation with an impedance condition prescribed on the boundary of a hole. The topological derivative is then used for numerical simulations for an inverse problem.

RÉSUMÉ

L'analyse de la sensibilité topologique fournit une expression asymptotique de la variation d'une fonctionnelle de forme par rapport à la taille d'un petit trou inséré à l'intérieur du domaine. Dans cet article une telle expression est obtenue pour l'équation de Helmholtz modifiée avec une condition d'impédance imposée sur le bord du trou. La dérivée topologique est alors utilisée pour des simulations numériques pour un problème inverse.

1. Introduction

The aim of the topological sensitivity analysis is to provide an asymptotic expansion of a shape functional with respect to the size of a small inclusion inserted inside the domain. To present the main idea, let us consider a domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) and a cost function $j(\Omega) = J(u_\Omega)$, where u_Ω is the solution to a given PDE defined over Ω. For a small parameter $\varepsilon > 0$, let Ω_ε be the domain obtained by removing a small part $\overline{x_0 + \varepsilon\omega}$ from Ω, where $x_0 \in \Omega$ and ω is a fixed bounded domain in R^d containing the origin, that is, $\Omega_\varepsilon = \Omega \backslash \overline{x_0 + \varepsilon\omega}$. In general, we have the following asymptotic expansion (as $\varepsilon \to 0^+$):

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http://dx.doi.org/10.1016/j.matpur.2014.07.003
0021-7824/

$$j(\Omega) - j(\Omega_\varepsilon) = f(\varepsilon)g(x_0) + o\big(f(\varepsilon)\big),$$

where $f(\varepsilon) > 0$ and $f(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. The function g is independent on ε and it is called topological gradient or topological derivative. To minimize the criterion j, one has to create holes at some points, where the topological gradient is negative.

The topological sensitivity analysis has been studied for different kinds of topology optimization problems: the elasticity case [14], the Poisson equation [15], the Navier–Stokes equation [4], the Helmholtz equation [28], the heat equation [6] and the wave equation [6]. For other works on topological sensitivity analysis, we refer the reader to [3,5,10,13,16,21,24–27,29,30].

In this paper, we apply the topological-shape sensitivity method to obtain the topological derivative for the modified Helmholtz equation under an impedance condition prescribed on the boundary of a hole.

The outline of this paper is as follows. The problem of interest is formulated in Section 2. In Section 3, we present some preliminaries including the adjoint method introduced in [24] and other useful results that will be used to establish our main result. The asymptotic analysis and the main result is presented in Section 4. Some numerical experiments are given in Section 5.

2. Problem formulation

Let Ω be a regular bounded domain in $\mathbb{R}^2$ with a regular boundary $\partial\Omega$. Let $u_\Omega \in H^1(\Omega)$ be the unique solution (weak solution) to the modified Helmholtz equation

$$-\Delta u_\Omega + a u_\Omega = 0 \quad \text{in } \Omega \tag{2.1}$$

with the Neumann boundary condition

$$\frac{\partial u_\Omega}{\partial n} = \zeta \quad \text{on } \partial\Omega, \tag{2.2}$$

where $a > 0$ is a constant and $\zeta \in H^{-1/2}(\partial\Omega)$.

Let $J : H^{1/2}(\partial\Omega) \to \mathbb{R}$ be a given differentiable mapping, and let

$$j(\Omega) := J(u_{\Omega|\partial\Omega}),$$

where $u_{\Omega|\partial\Omega}$ denotes the trace of $u_\Omega \in H^1(\Omega)$ on $\partial\Omega$. Let $p_\Omega \in H^1(\Omega)$ be the unique solution (weak solution) to the modified Helmholtz equation (2.1) with the boundary condition

$$\frac{\partial p_\Omega}{\partial n} = -DJ(u_{\Omega|\partial\Omega}) \quad \text{on } \partial\Omega, \tag{2.3}$$

where $DJ(u_{\Omega|\partial\Omega}) \in H^{-1/2}(\partial\Omega)$ denotes the derivative of J at the point $u_{\Omega|\partial\Omega}$.

For any sufficiently small parameter $\varepsilon > 0$, consider the perforated domain $\Omega_\varepsilon := \Omega \backslash \overline{B(x_0, \varepsilon)}$, where $x_0 \in \Omega$ and $\overline{B(x_0, \varepsilon)}$ is the closure of the open ball of center x_0 and radius ε. Possibly shifting the origin of the coordinate system, we assume for convenience that $x_0 = 0$. Let $u_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon)$ be the unique solution (weak solution) to the perturbed modified Helmholtz equation

$$-\Delta u_{\Omega_\varepsilon} + a u_{\Omega_\varepsilon} = 0 \quad \text{in } \Omega_\varepsilon \tag{2.4}$$

with an impedance condition on the boundary of the hole

$$u_{\Omega_\varepsilon} + \alpha \frac{\partial u_{\Omega_\varepsilon}}{\partial n} = 0 \quad \text{on } \partial B(x_0, \varepsilon) \tag{2.5}$$