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The horocycle flow at prime times

Peter Sarnak [a,*], Adrián Ubis [b]

[a] *Institute for Advanced Study, Einstein Road, Princeton, NJ 08540, USA*
[b] *Departamento de Matemáticas, Universidad Autónoma de Madrid, Madrid 28049, Spain*

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A B S T R A C T

We prove that the orbit of a non-periodic point at prime values of the horocycle flow in the modular surface is dense in a set of positive measure. For some special orbits we also prove that they are dense in the whole space—assuming the Ramanujan/Selberg Conjectures for $GL_2/\mathbb{Q}$. In the process, we derive an effective version of Dani's Theorem for the (discrete) horocycle flow.

R É S U M É

On démontre que l'orbite d'un point nonpériodique aux valeurs premières du flot horocycle sur la surface modulaire est dense dans un ensemble de mesure positive. En supposant les conjectures de Ramanujan–Selberg pour $GL_2/\mathbb{Q}$, on établit que certaines orbites spéciales sont denses dans tout l'espace. On en déduit une version effective du théorème de Dani pour le flot horocycle discret.

1. Introduction

If (X, T) is a dynamical system, for any $x \in X$ one can ask about the distribution of points $P_x = \{T^p x : p \text{ prime}\}$ in the orbit $\theta_x = \{T^n x : n \geq 1\}$. For example if X is finite then this is equivalent to Dirichlet's Theorem on primes in an arithmetic progression. If (X, T) is ergodic, Bourgain [4] shows that for almost all x, $T^p x$ with p prime, satisfies the Birkhoff Ergodic Theorem and hence is equidistributed. If (X, T) is 'chaotic', for example if it has a positive entropy then there may be many x's for which $T^p x$ is poorly distributed in $\overline{\theta_x}$. For example if $T : [0, 1] \to [0, 1]$ is the doubling map $x \mapsto 2x$ then one can construct an explicit (in terms of its binary expansion) ξ such that $\overline{\theta_\xi} = [0, 1]$ but $T^p \xi \to 0$ as $p \to \infty$.

* Corresponding author.
E-mail addresses: sarnak@math.princeton.edu (P. Sarnak), adrian.ubis@uam.es (A. Ubis).

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The setting in which one can hope for a regular behavior on restricting to primes is that of unipotent orbits in a homogeneous space. Let G be a connected Lie group, Γ a lattice in G and $u \in G$ an Ad_G unipotent element, then Ratner's Theorem [30] says that if $X = \Gamma\backslash G$ and $T : X \to X$ is given by

$$T(\Gamma g) = \Gamma g u, \tag{1}$$

then $\overline{\theta_x}$, with $x = \Gamma g$, is homogeneous and the orbit xu^n, $n = 1, 2, \ldots$ is equidistributed in $\overline{\theta_x}$ w.r.t. an algebraic measure $d\mu_x$. In the case that μ_x is the normalized volume measure $d\mu_G$ on X it is conjectured in [11] that $\overline{P_x} = X$ and in fact that xu^p, $p = 2, 3, 5, 7, 11, \ldots$ is equidistributed w.r.t. $d\mu_G$. Care should be taken in formulating this conjecture in the intermediate cases where $\overline{\theta_x}$ is not connected as there may be local congruence obstructions, but with the 'obvious' modifications this conjecture seems quite plausible. In intermediate cases where $\overline{\theta_x}$ is one of

(i) finite,
(ii) a connected circle or more generally a torus,
(iii) a connected nilmanifold $\Gamma\backslash N$,

$\overline{P_x}$ and the behavior of xu^p, $p = 2, 3, 5, \ldots$ is understood. Case (i) requires no further comment while for (ii) it follows from Vinogradov's work that the points are equidistributed w.r.t. dt, the volume measure on the torus. The same is true for (iii) as was shown recently by Green and Tao [12,13]; in order to prove this, apart from using Vinogradov's methods they had to control sums of the type $\sum_n e(\alpha n[\beta n])$, which are similar to Weyl sums but behave in a more complex way.

Our purpose in this paper is to examine this problem in the basic case of $X = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$. According to Hedlund [15], $\overline{\theta_x}$ is either finite, a closed horocycle of length l, $0 < l < \infty$, or is all X. The first two cases correspond to (i) and (ii). In the last case we say that x is *generic*. By a theorem of Dani [7] the orbit xu^n, $n = 1, 2, 3, \ldots$ is equidistributed in its closure w.r.t. one of the corresponding three types of algebraic measures. For $N \geq 1$ and $x \in X$ define the probability measure $\pi_{x,N}$ on X by

$$\pi_{x,N} := \frac{1}{\pi(N)} \sum_{p<N} \delta_{xu^p} \tag{2}$$

where for $\xi \in X$, δ_ξ is the delta mass at ξ and $\pi(N)$ is the number of primes less than N. We are interested in the weak limits ν_x of the $\pi_{x,N}$ as $N \to \infty$ (in the sense of integrating against continuous functions on the one-point compactification of X). If x is generic then the conjecture is equivalent to saying that any such ν_x is $d\mu_G$. One can also allow x, the initial point of the orbit, to vary with N in this analysis and in the measures in (2). Of special interest is the case $x = \Gamma g$

$$g = H_N := \begin{bmatrix} N^{-\frac{1}{2}} & 0 \\ 0 & N^{\frac{1}{2}} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

when $H_N u^j$, $0 \leq j \leq N-1$ is a periodic orbit for T of period N. These points are spread evenly on the unique closed horocycle in X whose length is N. They also comprise a large piece of the Hecke points in X corresponding to the Hecke correspondence of degree N

$$C_N = \left\{ \frac{1}{\sqrt{N}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad = N,\ a, d > 0,\ b \bmod d \right\}.$$

We can now state our main results. The first asserts that ν_x does not charge small sets with too much mass, that is ν_x is uniformly absolutely continuous with respect to $d\mu_G$.