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Nearly cloaking the elastic wave fields

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ABSTRACT

In this work, we develop a general mathematical framework on regularized approximate cloaking of elastic waves governed by the Lamé system via the approach of transformation elastodynamics. Our study is rather comprehensive. We first provide a rigorous justification of the transformation elastodynamics. Based on the blow-up-a-point construction, elastic material tensors for a perfect cloak are derived and shown to possess singularities. In order to avoid the singular structure, we propose to regularize the blow-up-a-point construction to be the blow-up-a-small-region construction. However, it is shown that without incorporating a suitable lossy layer, the regularized construction would fail due to resonant inclusions. In order to defeat the failure of the lossless construction, a properly designed lossy layer is introduced into the regularized cloaking construction. We derive sharp asymptotic estimates in assessing the cloaking performance. The proposed cloaking scheme is capable of nearly cloaking an arbitrary content with a high accuracy.

RÉSUMÉ

Dans cet article, on propose un cadre mathématique général pour la dissimulation d'une cible élastique dans le cadre des équations de Lamé via une approche de transformations élastodynamiques. Notre étude est exhaustive. Dans un premier temps on donne une justification rigoureuse des transformations élastodynamiques. En utilisant la construction d'un point singulier, on calcule les tenseurs élastiques nécessaires à l'invisibilité et on montre qu'ils possèent des singularités. Pour palier ce problème, on propose une méthode de régularisation. Cependant, on montre que sans l'incorporation d'une couche absorbante convenable, la structure régularisée présente des petites inclusions résonnantes qui la fragilise. Afin de contourner ce problème, on introduit une couche absorbante convenable lors de la construction. On quantifie les performances par des majorations asymptotiques optimales. La méthode proposée peut masquer quasi-complètement une cible arbitraire avec une bonne précision.

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1. Introduction

This paper concerns the cloaking of elastic waves. An elastic region is said to be cloaked if its content together with the cloak is "unseen" by the exterior elastic wave detections. In recent years, the study on elastic cloaking has gained growing interest in the physics literature (cf. [9,20–22,47,53,54]), much followed the development of transformation-optics cloaking of optical waves including the acoustic and electromagnetic waves. A proposal for cloaking for electrostatics using the invariance properties of the conductivity equation was pioneered in [26,27]. Blueprints for making objects invisible to electromagnetic (EM) waves were proposed in two articles in *Science* in 2006 [40,55]. The article by Pendry et al. uses the same transformation used in [26,27] while the work of Leonhardt uses a conformal mapping in two dimensions. The method based on the invariance properties of the equations modeling the optical wave phenomenon has been named *transformation optics* and has received a lot of attentions in the scientific community due to its significant practical importance. We refer to the survey articles [16,28,29,45] and the references therein for the theoretical and experimental progress on optical cloaking.

The Lamé system governing the elastic wave propagation also possesses a certain transformation property, in a more complicated manner than that for the optical wave equations. Using the transformation property, the transformation-elastodynamics approach can be developed for the construction of elastic cloaks, following a similar spirit to the transformation-optics construction of optical cloaks. In a rather heuristic way, an ideal invisibility cloak can be obtained by the blow-up-a-point construction proceeded as follows. One first selects a region Ω in the homogeneous space for constructing the cloaking device. Let $P \in \Omega$ be a single point and let F be a diffeomorphism which blows up P to a region D within Ω. Using transformation-elastodynamics, the ambient homogeneous medium around P is then 'compressed' via the push-forward to form the cloaking medium in $\Omega\backslash\overline{D}$, whereas the 'hole' D forms the cloaked region within which one can place the target object. The cloaking region $\Omega\backslash\overline{D}$ and the cloaked region D yield the cloaking device in the physical space, whereas the homogeneous background space containing the singular point P is referred to as the virtual space. Due to the transformation invariance of the elastic system, the exterior measurements corresponding to the cloaking device in the physical space are the same to those in the virtual space corresponding to a singular point. Intuitively speaking, the scattering information of the elastic cloak is 'hidden' in a singular point P.

However, the blow-up-a-point construction would yield server singularities for the cloaking elastic material tensors. Most of the physics literature accepts the singular structure and focuses more on the application side (cf. [9,20,21,53]). To our best knowledge, there is very little mathematical study on rigorously dealing with the singular elastic cloaking problem. On the other hand, there are a few mathematical works seriously dealing with the singular cloaking problems associated with the optical cloaks. Concurrently, there are two theoretical approaches in the literature: one approach is to accept the singularity and proposes to investigate the physically meaningful solutions, i.e. finite energy solutions, to the singular acoustic and electromagnetic wave equations (see [30,45]); the other approach is to regularize the singular ideal cloaking construction and investigate the near-invisibility instead; see [6,38] on the treatment of electrostatics, [4,7,37,41–43] on acoustics, and [8,10,11] on electromagnetism. In this work, we follow the latter approach to develop a general framework of constructing near-cloaks for elastic waves via the transformation-elastodynamics approach. Compared to the acoustic and electromagnetic cases, the elastic cloaking problem turns out to possess more complicated physical nature due to the coupling of shear and pressure waves that propagate at different speeds (see, e.g., [17–19,39]).

The present study on regularized approximate cloaking of elastic waves is rather comprehensive and includes several salient ingredients. First, we provide a rigorous justification of the transformation elastodynamics, which lacks in the physics literature. Particularly, we prove the well-posedness of the transformed Lamé system. This is presented in Section 2. In Section 3, we consider the elastic cloaking problem, and based on the blow-up-a-point transformation, we give the construction of an ideal elastic cloak and analyze the