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Existence and symmetry of least energy nodal solutions for Hamiltonian elliptic systems

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ABSTRACT

In this paper we prove existence of least energy nodal solutions for the Hamiltonian elliptic system with Hénon-type weights

$$-\Delta u = |x|^{\beta}|v|^{q-1}v, \quad -\Delta v = |x|^{\alpha}|u|^{p-1}u \quad \text{in } \Omega, \qquad u = v = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$, $\alpha, \beta \geq 0$ and the nonlinearities are superlinear and subcritical, namely

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}.$$

When Ω is either a ball or an annulus centred at the origin and $N \geq 2$, we show that these solutions display the so-called foliated Schwarz symmetry. It is natural to conjecture that these solutions are not radially symmetric. We provide such a symmetry breaking in a range of parameters where the solutions of the system behave like the solutions of a single equation. Our results on the above system are new even in the case of the Lane–Emden system (i.e. without weights). As far as we know, this is the first paper that contains results about least energy nodal solutions for strongly coupled elliptic systems and their symmetry properties.

RÉSUMÉ

Dans cet article, on démontre l'existence d'une solution nodale d'énergie minimale pour un système elliptique hamiltonien avec des poids de type Hénon

$$-\Delta u = |x|^{\beta}|v|^{q-1}v, \quad -\Delta v = |x|^{\alpha}|u|^{p-1}u \quad \text{in } \Omega, \qquad u = v = 0 \text{ on } \partial\Omega,$$

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où Ω est un domaine borné et régulier de $\mathbb{R}^N$, $N \geq 1$, $\alpha, \beta \geq 0$ et les non-linéarités sont superlinéaires et sous-critiques, c'est-à-dire

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}.$$

Lorsque Ω est une boule ou un anneau et $N \geq 2$, on montre que les solutions nodales d'énergie minimale possède la symétrie de Schwarz feuilletée. Il est naturel de conjecturer que ces solutions ne sont pas à symétries radiales. On démontre une telle brisure de symétrie dans une gamme des paramètres où les solutions du système se comportent comme les solutions d'une seule équation scalaire. Nos résultats sont nouveaux, déjà pour le système de Lane–Emden, c'est-à-dire sans poids. A notre connaissance, il s'agit du premier article qui présente des résultats concernant l'existence et les propriétés de symétrie de solutions nodales d'énergie minimale pour des systèmes elliptiques fortement couplés.

1. Introduction

We consider the Hamiltonian elliptic system with Hénon-type weights

$$-\Delta u = |x|^\beta |v|^{q-1} v, \quad -\Delta v = |x|^\alpha |u|^{p-1} u \quad \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, and $\alpha, \beta \geqslant 0$. We consider superlinear and subcritical nonlinearities, namely

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}. \tag{H}$$

Observe that the first condition is also equivalent to $pq > 1$.

The system (1.1) is strongly coupled in the sense that $u \equiv 0$ if and only if $v \equiv 0$. Moreover, u changes sign if and only if v changes sign.

We recall that a strong solution to this problem corresponds to a pair (u, v) with

$$u \in W^{2,(q+1)/q}(\Omega) \cap W_0^{1,(q+1)/q}(\Omega), \qquad v \in W^{2,(p+1)/p}(\Omega) \cap W_0^{1,(p+1)/p}(\Omega)$$

satisfying the system in (1.1) for a.e. $x \in \Omega$. By using a bootstrap method (see [39, Theorem 1(a)], [9, Proposition 2.1] and [25, Theorem 1.1]), it can be shown that strong solutions are actually classical solutions.

Consider the energy functional

$$E(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx - \frac{1}{p+1} \int_\Omega |x|^\alpha |u|^{p+1} \, dx - \frac{1}{q+1} \int_\Omega |x|^\beta |v|^{q+1} \, dx, \tag{1.2}$$

which is well defined for strong solutions thanks to assumption (H).

One can use various variational settings to deal with the system (1.1), see for instance the surveys [11,22,36]. Once the existence of at least one critical point is proved, a natural question is that of the existence of a least energy one, by which we mean a critical point at the level

$$c = \inf\{E(u, v) : \ (u, v) \text{ is a nonzero strong solution of (1.1)}\}.$$

The solutions at this energy are usually referred to as ground state solutions and in many problems, they are of special interest. In our setting the existence of such solutions is clear and rely on a simple compactness