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Nonlinear Korn inequalities

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ABSTRACT

Let Ω be a bounded and connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary Γ, the set Ω being locally on the same side of Γ, and let $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{R}^n$ and $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^n$ be two smooth enough "deformations" of the set $\overline{\Omega}$. Then the classical Korn inequality asserts that, when $\boldsymbol{\Theta} = \boldsymbol{id}$, there exists a constant c such that

$$\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \leq c\Big(\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^T\|_{\mathbb{L}^2(\Omega)}\Big) \text{ for all } \boldsymbol{v} \in \mathbf{H}^1(\Omega),$$

where $\boldsymbol{v} := (\boldsymbol{\Phi} - \boldsymbol{id}) : \overline{\Omega} \to \mathbb{R}^n$ denotes the corresponding "displacement" vector field, and where the symmetric tensor field $\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^T : \overline{\Omega} \to \mathbb{S}^n$ is nothing but the linear part with respect to $\boldsymbol{v}$ of the difference between the metric tensor fields $\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi}$ and $\boldsymbol{I}$ that respectively correspond to the deformations $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta} = \boldsymbol{id}$.
Assume now that the identity mapping $\boldsymbol{id}$ is replaced by a more general orientation-preserving immersion $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$. We then show in particular that, given any $1 < p < \infty$ and any $q \in \mathbb{R}$ such that $\max\{1, \frac{p}{2}\} \leq q \leq p$, there exists a constant $C = C(p, q, \boldsymbol{\Theta})$ such that

$$\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{q/p}_{\mathbb{L}^q(\Omega)}\Big)$$

for all $\boldsymbol{\Phi} \in \mathbf{W}^{1,2q}(\Omega)$ that satisfy $\det \boldsymbol{\nabla\Phi} > 0$ almost everywhere in Ω. Such an inequality thus constitutes an instance of a "nonlinear Korn inequality", in the sense that the symmetric tensor field $\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} : \overline{\Omega} \to \mathbb{S}^n$ appearing in its right-hand side is now the exact difference between the metric tensor fields corresponding to the deformations $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$.
We also show that, like in the linear case, an analogous nonlinear Korn inequality holds, but without the norm $\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{L}^p(\Omega)}$ in its right-hand side, if the difference $\boldsymbol{\Phi} - \boldsymbol{\Theta}$ vanishes on a subset Γ_0 of Γ with $d\Gamma$-meas $\Gamma_0 > 0$.
The key to providing such nonlinear Korn inequalities is a generalization of the landmark "geometric rigidity lemma in $\mathbf{H}^1(\Omega)$" established in 2002 by G. Friesecke, R.D. James, and S. Müller, as later extended to $\mathbf{W}^{1,p}(\Omega)$ by S. Conti.

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R É S U M É

Soit Ω un ouvert borné connexe de $\mathbb{R}^n$ de frontière Γ lipschitzienne, l'ensemble Ω étant localement d'un seul coté de Γ, et soit $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{R}^n$ et $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^n$ deux "déformations" suffisamment régulières de l'ensemble $\overline{\Omega}$. Alors l'inégalité de Korn classique exprime que, lorsque $\boldsymbol{\Theta} = \boldsymbol{id}$, il existe une constante c telle que

$$\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \leq c\Big(\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^T\|_{\mathbb{L}^2(\Omega)}\Big) \text{ pour tout } \boldsymbol{v} \in \mathbf{H}^1(\Omega),$$

où $\boldsymbol{v} := (\boldsymbol{\Phi} - \boldsymbol{id}) : \overline{\Omega} \to \mathbb{R}^n$ represente le champ de "déplacements" correspondant, et où le champ de tenseurs symétriques $\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^T : \overline{\Omega} \to \mathbb{S}^n$ n'est autre que la partie linéaire en $\boldsymbol{v}$ de la différence entre les champs de tenseurs métriques $\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi}$ et $\boldsymbol{I}$, qui correspondent respectivement aux déformations $\boldsymbol{\Phi}$ et $\boldsymbol{\Theta} = \boldsymbol{id}$.
Supposons maintenant que l'application identique $\boldsymbol{id}$ est remplacée par une immersion plus générale $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ qui préserve l'orientation. Alors on montre en particulier que, pour tout $1 < p < \infty$ et pour tout $q \in \mathbb{R}$ tels que $\max\{1, \frac{p}{2}\} \leq q \leq p$, il existe une constante $C = C(p, q, \boldsymbol{\Theta})$ telle que

$$\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|_{\mathbb{L}^q(\Omega)}^{q/p}\Big)$$

pour tout $\boldsymbol{\Phi} \in \mathbf{W}^{1,2q}(\Omega)$ satisfaisant $\det \boldsymbol{\nabla\Phi} > 0$ presque partout dans Ω. Une telle inégalité constitue donc un exemple d'"inégalité de Korn non linéaire", au sens que le champ de tenseurs symétriques $\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} : \overline{\Omega} \to \mathbb{S}^n$ apparaissant dans son membre de droite est maintenant la différence exacte entre les champs de tenseurs métriques correspondant aux déformations $\boldsymbol{\Phi}$ et $\boldsymbol{\Theta}$.
On montre également que, comme dans le cas linéaire, on peut établir une inégalité de Korn non linéaire analogue, mais sans la norme $\|\boldsymbol{\Phi} - \boldsymbol{\Theta}\|_{\mathbf{L}^p(\Omega)}$ dans son membre de droite, si la différence $\boldsymbol{\Phi} - \boldsymbol{\Theta}$ s'annule sur une partie Γ_0 de Γ telle que $d\Gamma$-mes $\Gamma_0 > 0$.
La clef pour établir de telles inégalités de Korn non linéaires est une généralisation du remarquable "lemme de rigidité géométrique dans $\mathbf{H}^1(\Omega)$" établi en 2002 par G. Friesecke, R.D. James, et S. Müller, tel qu'il a été ensuite étendu à $\mathbf{W}^{1,p}(\Omega)$ par S. Conti.

1. Introduction

All the notations and definitions appearing in this introduction are defined and explained in the next section.

In a landmark contribution, Friesecke, James and Müller [9] have shown that, given any domain Ω in $\mathbb{R}^n$, there exists a constant c_1 such that, for all mappings $\boldsymbol{\Phi} \in \mathbf{H}^1(\Omega)$,

$$\inf_{\boldsymbol{R}\in\mathbb{O}^n_+} \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\|_{\mathbb{L}^2(\Omega)} \leq c_1 \| \inf_{\boldsymbol{R}\in\mathbb{O}^n_+} |\boldsymbol{\nabla\Phi} - \boldsymbol{R}|\|_{\mathbb{L}^2(\Omega)}.$$

This fundamental "*geometric rigidity lemma*" has crucial applications in nonlinear plate and shell theories, in particular for proving that the three-dimensional model of nonlinear elasticity Γ-converges towards a two-dimensional nonlinear flexural plate, or shell, model as the thickness of the plate, or shell, goes to zero; cf. [9–11].

A first generalization of this geometric rigidity lemma was given by Conti [7], who showed that the $L^2(\Omega)$-norms can be replaced by any $L^p(\Omega)$-norms, $1 < p < \infty$, in both sides of the inequality (with a constant now depending on p).

In this paper, we further generalize this lemma, by showing that the identity mapping $\boldsymbol{id}$, which implicitly appears through the factor $\boldsymbol{I} = \boldsymbol{\nabla id}$ to the right of the matrices $\boldsymbol{R}$, can be replaced by an arbitrary