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A variational approach to second order mean field games with density constraints: The stationary case

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ABSTRACT

In this paper we study second order stationary Mean Field Game systems under density constraints on a bounded domain $\Omega \subset \mathbb{R}^d$. We show the existence of weak solutions for power-like Hamiltonians with arbitrary order of growth. Our strategy is a variational one, i.e. we obtain the Mean Field Game system as the optimality condition of a convex optimization problem, which has a solution. When the Hamiltonian has a growth of order $q' \in]1, d/(d-1)[$, the solution of the optimization problem is continuous which implies that the problem constraints are qualified. Using this fact and the computation of the subdifferential of a convex functional introduced by Benamou and Brenier (see [1]), we prove the existence of a solution of the MFG system. In the case where the Hamiltonian has a growth of order $q' \geq d/(d-1)$, the previous arguments do not apply and we prove the existence by means of an approximation argument.

RÉSUMÉ

Dans cet article on étudie des systèmes de jeux à champ moyen sous contrainte de densité sur un domaine borné $\Omega \subset \mathbb{R}^d$. On démontre l'existence de solutions faibles pour des hamiltoniens de type puissance avec ordre de croissance arbitraire. Notre stratégie est variationnelle, on obtient le système de jeux à champ moyen comme condition d'optimalité d'un problème convexe, lequel a une solution. Quand l'hamiltonien a un ordre de croissance $q' \in]1, d/(d-1)[$, la solution du problème d'optimisation est continue, ce qui implique que les contraintes du problème sont qualifiées. En utilisant cette proprieté et le calcul du sous-différentiel d'une fonctionnelle convexe introduite par Benamou et Brenier (voir [1]), on démontre l'existence d'une solution du système MFG. Dans les cas où l'hamiltonien a un ordre de croissance $q' \geq d/(d-1)$, les arguments précédents ne sont pas applicables et on montre l'existence en utilisant un argument d'approximation.

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1. Introduction

The theory of Mean Field Games (shortly MFG in the sequel) was introduced recently and simultaneously by J.-M. Lasry and P.-L. Lions [2–4] and M. Huang, R.P. Malhamé and P.E. Caines (see [5]). The main objective of the MFG theory is the study of the limit behavior of Nash equilibria for symmetric differential games with a very large number of "small" players. In its simplest form, as the number of players tends to infinity, limits of Nash equilibria can be characterized in terms of the solution of the following coupled PDE system:

$$\begin{cases} -\partial_t u(t,x) - \nu\Delta u(t,x) + H(x,\nabla u(t,x)) = f[m(t)](x) & \text{in } (0,T]\times\mathbb{R}^d, \\ \partial_t m(t,x) - \nu\Delta m(t,x) - \operatorname{div}(\nabla_p H(x,\nabla u(t,x))m(t,x)) = 0 & \text{in } (0,T]\times\mathbb{R}^d, \\ m(0,x) = m_0, \quad u(T,x) = g(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{MFG}$$

where $H(x,\cdot)$ is convex. The Hamilton–Jacobi–Bellman (HJB) equation in (MFG) characterizes the value function $u[m]$ associated to a stochastic optimal control problem solved by a typical player whose cost function depends at each time t on the distribution $m(t,\cdot)$ of the other agents. We remark that this interaction can be global, e.g. if $f[m(t,\cdot)](x)$ is a convolution of $m(t,\cdot)$ with another function, or local, i.e. when $f[m(t)](x)$ can be identified to a function $f(x,m(t,x))$. The Fokker–Planck equation (FP) in (MFG) describes the evolution $m[u]$ of the initial distribution m_0 when all the agents follow the optimal feedback strategy computed by the typical agent. We refer the reader to the original papers [2–4] and the lectures [6] for more details on the modeling and the relation with the system (MFG). See also [7,8] for a survey on the subject.

For local couplings $f(\cdot,m)$, system (MFG) can be obtained (at least formally) as the optimality condition of problem

$$\begin{aligned} &\min \int_0^T \int_{\mathbb{R}^d} \left\{ m(t,x) L\left(x, -\frac{w(t,x)}{m(t,x)}\right) + F(x,m(t,x)) \right\} \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} g(x)m(T,x)\,\mathrm{d}x, \\ &\text{s.t. } \partial_t m - \nu\Delta m + \operatorname{div}(w) = 0, \quad m(0,x) = m_0, \end{aligned} \tag{1.1}$$

with $F(x,m) := \int_0^m f(x,m')\,\mathrm{d}m'$, $L(x,v) := H^*(x,v)$ (where the Fenchel conjugate $H^*(x,v)$ is calculated on the second variable of H) and $m_0 \in L^\infty(\mathbb{R}^d)$ satisfying that $m_0 \geq 0$ and $\int_{\mathbb{R}^d} m_0\,\mathrm{d}x = 1$. This type of approach, including also the degenerate first order case ($\nu = 0$), has been studied extensively in the last years in a series of papers [9–12]. The optimization problem above recalls the so-called Benamou–Brenier formulation of the 2-Wasserstein distance between two probability measures, which gives a fluid mechanical or dynamical interpretation of the Monge–Kantorovich optimal transportation problem (see [1,13]). We refer the reader to [14,15] and the recent work [16] for some optimization methods to solve numerically (MFG) based on the formulation (1.1).

With a well-chosen time-averaging procedure, one can introduce stationary MFG systems as an ergodic limit of time dependent ones (see [17,18]),

$$\begin{cases} -\nu\Delta u(x) + H(x,\nabla u(x)) - \lambda = f(x,m(x)) & \text{in } \mathbb{R}^d \\ -\nu\Delta m(x) - \operatorname{div}(\nabla_p H(x,\nabla u(x))m(x)) = 0 & \text{in } \mathbb{R}^d, \\ \int_{\mathbb{R}^d} m(x)\,\mathrm{d}x = 1, \quad \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 0, \quad m \geq 0. \end{cases} \tag{MFG_∞}$$