J. Math. Pures Appl. 104 (2015) 1160–1189

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Uniqueness and stability results for an inverse spectral problem in a periodic waveguide

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ARTICLE INFO

Article history:
Received 1 June 2015
Available online 7 September 2015

MSC:
primary 35R30, 35J10
secondary 35P99

Keywords:
Inverse spectral problem
Stability
Uniqueness
Periodic waveguide
Schrödinger operator
Periodic potential

ABSTRACT

Let $\Omega := \omega \times \mathbb{R}$ where $\omega \subset \mathbb{R}^2$ is a bounded domain, and let $V : \Omega \to \mathbb{R}$ be a bounded potential which is 2π-periodic in the variable $x_3 \in \mathbb{R}$. We study the inverse problem consisting in the determination of V, through the boundary spectral data of the operator $u \mapsto Au := -\Delta u + Vu$, acting on $L^2(\omega \times (0, 2\pi))$, with quasi-periodic and Dirichlet boundary conditions. More precisely we show that if for $j = 1, 2$ two potentials V_j are given so that $\|V_j\|_\infty \leq R$, and if we denote by $(\lambda_{j,k})_k$ the eigenvalues of the operators A_j (that is the operator A with $V := V_j$), then for a constant $c > 0$, depending on ω and $R > 0$, we have $\|\mathcal{F}((V_1 - V_2)1_{\omega\times(0,2\pi)})\|_\infty \leq c \limsup_{k\to\infty} |\lambda_{1,k} - \lambda_{2,k}|$, provided that $\sum_{k\geq 1} \|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\omega\times[0,2\pi])} < \infty$, where $\psi_{j,k} := \partial\phi_{j,k}/\partial\mathbf{n}$ (here $\mathcal{F}$ denotes the Fourier transform). The arguments developed here may be applied to other spectral inverse problems, and similar results can be obtained.

RÉSUMÉ

Soit $\Omega = \omega \times \mathbb{R}$ où $\omega \subset \mathbb{R}^2$ est un domaine borné, et soit $V : \Omega \to \mathbb{R}$ un potentiel borné 2π-periodique en la variable $x_3 \in \mathbb{R}$. On étudie le problème inverse consistant à déterminer V grâce aux données spectrales sur le bord de l'opérateur $u \mapsto Au := -\Delta u + Vu$, agissant sur $L^2(\omega \times (0, 2\pi))$, avec des conditions au bord de Dirichlet et quasi-periodiques. Plus précisément, on démontre que si pour $j = 1, 2$ deux potentiels V_j sont donnés tels que $\|V_j\|_\infty \leq R$, et si on désigne par $(\lambda_{j,k})_k$ les valeurs propres associées aux opérateurs A_j (c'est-à-dire l'opérateur A avec $V := V_j$), alors pour une constante $c > 0$, dépendant de ω et de $R > 0$, on an $\|\mathcal{F}((V_1 - V_2)1_{\omega\times(0,2\pi)})\|_\infty \leq c \limsup_{k\to\infty} |\lambda_{1,k} - \lambda_{2,k}|$, pourvu que $\sum_{k\geq 1} \|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\omega\times[0,2\pi])} < \infty$, où $\psi_{j,k} := \partial\phi_{j,k}/\partial\mathbf{n}$ (ici $\mathcal{F}$ est la transformation de Fourier). Les arguments peuvent être utilisés pour d'autres problèmes inverses spectraux, des résultats similaires peuvent être obtenus.

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http://dx.doi.org/10.1016/j.matpur.2015.09.002
0021-7824/

1. Introduction

In the present paper we study two related inverse spectral problems in which a potential is identified through an incomplete boundary spectral data.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain. On the one hand set

$$Y := \omega \times (0, 2\pi) \qquad \text{and} \qquad \Gamma := \partial\omega \times [0, 2\pi], \tag{1.1}$$

and on the other hand consider an infinite waveguide Ω with

$$\Omega := \omega \times \mathbb{R} \qquad \text{and} \qquad \partial\Omega = \partial\omega \times \mathbb{R}. \tag{1.2}$$

We may assume, without loss of generality, that the cross section ω of the waveguide contains the origin $0_{\mathbb{R}^2}$ of $\mathbb{R}^2$. For simplicity we assume that ω is a $\mathcal{C}^{1,1}$ domain. Nevertheless, with some additional arguments most of the results of this paper (Theorems 1.1–1.3) can be treated when ω is assumed to be only a Lipschitz domain. For the sake of brevity of notations, we write $x = (x', x_3)$ with $x' = (x_1, x_2) \in \omega$ for every $x = (x_1, x_2, x_3) \in \Omega$.

The main problem we study, and whose solution is a consequence of a result presented a few lines below, concerns an inverse spectral problem in a waveguide given by Ω. We consider a real valued electric potential $V \in L^\infty(\Omega; \mathbb{R})$, which is 2π-periodic with respect to the infinite variable x_3. Namely, we assume that V satisfies

$$V(x', x_3 + 2\pi) = V(x', x_3), \quad \forall\, x_3 \in \mathbb{R}, \tag{1.3}$$

and then we define the self-adjoint operator $(A, D(A))$ acting in $L^2(\Omega)$ by

$$Au := -\Delta u + Vu, \quad \text{for } \ u \in D(A) \tag{1.4}$$

with its domain

$$D(A) := \left\{ u \in H_0^1(\Omega)\ ;\ -\Delta u + Vu \in L^2(\Omega) \right\}. \tag{1.5}$$

We are interested in the problem of determining V from the partial knowledge of the spectral data associated with A. However, the operator $(A, D(A))$ being self-adjoint and its resolvent not being compact, it may have a continuous spectrum contained in an interval of type $[\lambda_*, +\infty)$: thus in the first place one should state precisely what is meant by an inverse spectral problem. To make this statement more precise, we are going to recall the definition of the (full) spectral data associated with the operator A, but before doing so we state another result closely related to the above problem.

This result concerns the following inverse spectral problem: let Y be as in (1.1) and consider a real valued potential $V \in L^\infty(Y)$ and, for a given fixed $\theta \in [0, 2\pi)$, let $(\lambda_k(\theta), \varphi_{\theta,k})_{k\geq 1}$ be the eigenvalues and normalized eigenfunctions of the realization of the operator $-\Delta + V$ with quasi-periodic and Dirichlet boundary conditions, more precisely those eigenvalues and eigenfunctions given by

$$\begin{cases} -\Delta\varphi_{\theta,k} + V\varphi_{\theta,k} = \lambda_k(\theta)\varphi_{\theta,k} & \text{in } Y, \\ \varphi_{\theta,k}(\sigma) = 0, & \sigma \in \Gamma, \\ \varphi_{\theta,k}(x', 2\pi) = \mathrm{e}^{\mathrm{i}\theta}\varphi_{\theta,k}(x', 0), & x' \in \omega, \\ \partial_3\varphi_{\theta,k}(x', 2\pi) = \mathrm{e}^{\mathrm{i}\theta}\partial_3\varphi_{\theta,k}(x', 0), & x' \in \omega. \end{cases} \tag{1.6}$$

Then we show that if $N \geq 1$ is a given integer, knowledge of

$$\lambda_j(\theta),\ 1_\Gamma \frac{\partial\varphi_{\theta,j}}{\partial\mathbf{n}} \quad \text{for } \ j \geq N+1,$$