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Reducibility of one-dimensional quasi-periodic Schrödinger equations

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ABSTRACT

Consider the time-dependent linear Schrödinger equation

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n + \delta \sum_{m\in\mathbb{Z}} a_{mn}(\theta + \xi t)q_m, \quad n \in \mathbb{Z},$$

where V is a nonconstant real-analytic function on $\mathbb{T}$, ω satisfies a certain Diophantine condition and $a_{mn}(\theta)$ is real-analytic on $\mathbb{T}^b$, $b \in \mathbb{Z}_+$, decaying with $|m|$ and $|n|$. We prove that, if ϵ and δ are sufficiently small, then for a.e. $x \in \mathbb{T}$ and "most" frequency vectors $\xi \in \mathbb{T}^b$, it can be reduced to an autonomous equation. Moreover, for this non-autonomous system, "dynamical localization" is maintained in a quasi-periodic time-dependent way.

RÉSUMÉ

On considère l'équation de Schrödinger linéaire qui dépend du temps

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n + \delta \sum_{m\in\mathbb{Z}} a_{mn}(\theta + \xi t)q_m, \quad n \in \mathbb{Z},$$

où V est une fonction analytique réelle non-constante sur $\mathbb{T}$, ω satisfait une certaine condition diophantienne et $a_{mn}(\theta)$ est analytique réelle sur $\mathbb{T}^b$, $b \in \mathbb{Z}_+$, qui déscroît avec $|m|$ et $|n|$. On démontre que, si ϵ et δ sont suffisamment petits, pour presque tous les $x \in \mathbb{T}$ et « la plupart » des vecteurs de fréquence $\xi \in \mathbb{T}^b$, elle peut être réduite à une équation autonome.
En outre, pour ce système non-autonome, la « localisation dynamique » est maintenue d'une manière quasi-périodique en fonction du temps.

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1. Introduction and Main results

We consider the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n + \delta \sum_{m\in\mathbb{Z}} a_{mn}(\theta + \xi t)q_m, \quad n \in \mathbb{Z}, \tag{1}$$

where V is a nonconstant real-analytic function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\omega \in \mathbb{R}$ is a Diophantine number, and for each $m, n \in \mathbb{Z}$, $a_{mn} : \mathbb{T}^b \to \mathbb{R}$ is analytic in a complex neighborhood $|\mathrm{Im}\theta| < r \le 1$ of the b-torus $\mathbb{T}^b$, satisfying

$$\sup_{|\mathrm{Im}\theta|<r} |a_{mn}(\theta)| < e^{-\rho \max\{|m|, |n|\}}, \quad \rho > 0.$$

We are going to prove that, for ϵ and δ small enough, Eq. (1) can be reduced to a constant coefficient equation (independent of t) for "most" value of the parameter ξ, with the corresponding solutions well localized in space all the time. It is stated in the following theorem:

Theorem 1. *There exists a sufficiently small $\epsilon_* = \epsilon_*(V, \omega, r, \rho)$, such that if $0 < \epsilon, \delta < \epsilon_*$, then for a.e. $x \in \mathbb{T}$, one can find a Cantor set $\mathcal{O}_{\epsilon,\delta} = \mathcal{O}_{\epsilon,\delta}(x) \subset \mathbb{T}^b$ with*

$$\mathrm{Leb}(\mathbb{T}^b \setminus \mathcal{O}_{\epsilon,\delta}) \to 0 \quad as \quad \epsilon, \delta \to 0,$$

such that for each $\xi \in \mathcal{O}_{\epsilon,\delta}$ and $\theta \in \mathbb{T}^b$, Eq. (1) can be analytically reduced to an autonomous equation.

Moreover, given any initial datum $q(0)$ satisfying $|q_n(0)| < ce^{-\rho|n|}$ for some constant $c > 0$, the solution $q(t)$ to Eq. (1) with $\xi \in \mathcal{O}_{\epsilon,\delta}$ satisfies that for any fixed $d > 0$,

$$\sup_t \sum_{n\in\mathbb{Z}} n^{2d} |q_n(t)|^2 < \infty.$$

Remark 1.1. The behavior of solutions for a dynamical equation in the last statement of Theorem 1 is called dynamical localization.

Eq. (1) is a perturbation of an autonomous quasi-periodic Schrödinger equation, whose behavior is determined by the spectral property of the operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined as

$$(Tq)_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n.$$

As shown in [13], for ϵ small enough, the spectrum of this operator is pure point for a.e. $x \in \mathbb{T}$. More precisely, it can be "almost block-diagonalized", which is presented by a KAM scheme (see Proposition 1 for details). The readers can also refer to [6,10,16,24,25,28] for other works on the pure point spectrum and localization of quasi-periodic Schrödinger operators, and see [12,22,23] for more about dynamical localization.

It is necessary to mention that the KAM theory has been well adapted to Hamiltonian PDE's, especially in the continuous case. Many well-known works have been done for construction of time quasi-periodic solutions (e.g., [3,15,17–19]), for reducibility of non-autonomous equations (e.g., [1,14]), and for growth of Sobolev norms (e.g., [2,4,5,29,30]).

However, the KAM technique is not widely applied to the discrete models, especially the case of dense normal frequencies. A successful application is the model

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\alpha)q_n + |q_n|^2 q_n, \quad n \in \mathbb{Z}, \tag{2}$$