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A characterization of compactness for singular integrals

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ABSTRACT

We prove a $T(1)$ theorem to completely characterize compact Calderón–Zygmund operators. The result provides sufficient and necessary conditions for the compactness of singular integral operators acting on $L^p(\mathbb{R})$ with $1 < p < \infty$.

RÉSUMÉ

On démontre un théorème $T(1)$ caractérisant complètement la compacité des opérateurs de Calderón–Zygmund. Le résultat donne des conditions nécessaires et suffisantes pour que des opérateurs intégraux singulieres soient compact sur $L^p(\mathbb{R})$, $1 < p < \infty$.

1. Introduction

The theory of singular integrals started with the study of singular convolution operators such as the Hilbert and Riesz transforms. David and Journé's famous $T(1)$ theorem [1] marked significant progress in the theory, characterizing the boundedness of a larger class of singular integral operators, including operators of non-convolution type. In contrast with convolution operators, some of these operators are compact. Well known examples include certain Hankel operators [2], commutators of singular integrals and multiplication operators [3], and layer potential operators [4]. The study of compactness of singular integral operators has been an area of active research, and continues to be so today with a variety of applications, many in the field of elliptic partial differential equations [5].

The purpose of this paper is to develop a general theory of compactness for a large class of singular integrals that, in the spirit of David and Journé, describes when a Calderón–Zygmund operator is compact in terms of its action on special families of functions. We actually present a new $T(1)$ theorem which characterizes the compactness of singular integral operators, in analogy with the characterization of their

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boundedness given by the classical $T(1)$ theorem. More precisely, we show that a singular integral operator T is compact on $L^p(\mathbb{R})$ with $1 < p < \infty$ if and only if its kernel satisfies the definition of what we call a compact Calderón–Zygmund kernel, the operator satisfies a new property of *weak compactness*, analogue to the classical *weak boundedness*, and the functions $T(1)$ and $T^*(1)$ belong to the space $\mathrm{CMO}(\mathbb{R})$, the appropriate substitute of $\mathrm{BMO}(\mathbb{R})$.

Our hypotheses impose additional smoothness bounds on the kernel of T, beyond it being of Calderón–Zygmund type. However, it is important to notice that these additional smoothness and decay conditions are, on the one side, necessary and, on the other side, they can be of arbitrary size, and hence fully singular kernels are within the scope of our theorem.

The paper is structured as follows. In Section 2 we give necessary definitions and state our main result, Theorem 2.21. In Section 3 we prove the necessity of the hypotheses of Theorem 2.21: in Proposition 3.2 we show that compact Calderón–Zygmund operators are associated with compact Calderón–Zygmund kernels; in Proposition 3.5 we prove necessity of the weak compactness condition; and in Proposition 3.12 we show the membership of $T(1)$ and $T^*(1)$ in $\mathrm{CMO}(\mathbb{R})$. All remaining sections are devoted to prove their sufficiency. In Section 4 we prove a fundamental lemma describing the action of the operator on bump functions. In Section 5 we demonstrate compactness on L^p, $1 < p < \infty$, under special cancellation conditions. Finally, in Section 6 we construct the paraproducts which allow to show compactness in full generality.

We would like to highlight here two surprising facts. The first one is the lack of use of the kernel decay of a standard Calderón–Zygmund kernels in our calculations. As proved in Lemmata 2.4 and 3.4, this is because the kernel smoothness and the weak boundedness condition imply the decay condition. The second one is that, unlike other compact operators like Hilbert–Schmidt operators for instance, compact Calderón–Zygmund operators are associated with kernels that satisfy precise pointwise decay estimates in the directions perpendicular and parallel to the diagonal (see Proposition 3.2).

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2. Definitions and statement of the main result

2.1. Compact Calderón–Zygmund kernel

Definition 2.1. Three bounded functions $L, S, D : [0, \infty) \to [0, \infty)$ constitute a set of admissible functions if the following limits hold:

$$\lim_{x\to\infty} L(x) = \lim_{x\to 0} S(x) = \lim_{x\to\infty} D(x) = 0. \tag{1}$$

Remark 2.2. Since any fixed dilation of an admissible function $L_\lambda(x) = L(\lambda^{-1}x)$ is again admissible, we will often omit all universal constants appearing in the argument of these functions.

Definition 2.3. Let Δ be the diagonal of $\mathbb{R}^2$. A function $K : (\mathbb{R}^2 \setminus \Delta) \to \mathbb{C}$ is called a compact Calderón–Zygmund kernel if it is bounded on compact sets of $\mathbb{R}^2 \setminus \Delta$ and there exist $0 < \delta \leq 1$, $C > 0$ and L, S, D admissible functions such that

$$|K(t, x) - K(t', x')| \leq C \frac{(|t - t'| + |x - x'|)^\delta}{|t - x|^{1+\delta}} L(|t - x|) S(|t - x|) D(|t + x|),$$

whenever $2(|t - t'| + |x - x'|) < |t - x|$.