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Regularity in Monge's mass transfer problem ☆

Qi-Rui Li [a], Filippo Santambrogio [b,*], Xu-Jia Wang [a]

[a] *Centre for Mathematics and Its Applications, Australian National University, Canberra, ACT 0200, Australia*
[b] *Laboratoire de Mathematiques d'Orsay, Université Paris-Sud, 91405 Orsay cedex, France*

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ABSTRACT

In this paper, we study the regularity of optimal mappings in Monge's mass transfer problem. Using the approximation to Monge's cost function $c(x,y)=|x-y|$ through the costs $c_\varepsilon(x,y)=\sqrt{\varepsilon^2+|x-y|^2}$, we consider the optimal mappings T_ε for these costs, and we prove that the eigenvalues of the Jacobian matrix DT_ε, which are all positive, are locally uniformly bounded. By an example we prove that T_ε is in general not uniformly Lipschitz continuous as $\varepsilon \to 0$, even if the mass distributions are positive and smooth, and the domains are c-convex.

RÉSUMÉ

Dans cet article, on étudie la régularité des transports optimaux pour le problème de Monge. En utilisant l'approximation du coût de Monge $c(x,y)=|x-y|$ par des coûts $c_\varepsilon(x,y)=\sqrt{\varepsilon^2+|x-y|^2}$, on considère les transports optimaux T_ε, et on démontre que les valeurs propres de la matrice jacobienne DT_ε, qui sont toutes positives, sont localement uniformément bornées. À l'aide d'un exemple on démontre que T_ε n'est pas en général uniformément Lipschitz lorsque $\varepsilon \to 0$, même si les distributions de masse sont lisses et positives sur des domaines c-convexes.

1. Introduction

The Monge mass transfer problem consists in finding an optimal mapping from one mass distribution to another one such that the total cost is minimized among all measure preserving mappings. This problem was first proposed by Monge [27] and has been studied by many authors in the last two hundred years: among the main achievements in the 20th century we cite [21] and [16].

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* Corresponding author.
E-mail addresses: qi-rui.li@anu.edu.au (Q.-R. Li), filippo.santambrogio@math.u-psud.fr (F. Santambrogio), xu-jia.wang@anu.edu.au (X.-J. Wang).

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In Monge's problem, the cost of moving a mass from point x to point y is proportional to the distance $|x-y|$, namely the cost function is given by

$$c_0(x,y) = |x-y|. \tag{1.1}$$

This is a natural cost function. In the last two decades, due to a range of applications, the optimal transportation for more general cost functions has been a subject of extensive studies. In order to present the framework more precisely, let Ω and Ω^* be two bounded domains in the Euclidean space $\mathbb{R}^n$, and let f and g be two densities in Ω and Ω^* respectively, satisfying the mass balance condition

$$\int_\Omega f(x)\,dx = \int_{\Omega^*} g(y)\,dy. \tag{1.2}$$

Let c be a smooth cost function defined on $\Omega \times \Omega^*$.

The problem consists in finding a map $T : \Omega \to \Omega^*$ which solves

$$\min \int c\big(x, T(x)\big) f(x)\,dx: \quad T_\# f = g,$$

where the last condition reads "the image measure of f through T is g" and means $\int_A g(y)\,dy = \int_{T^{-1}(A)} f(x)\,dx$ for all subsets $A \subset \Omega^*$.

The existence and uniqueness of optimal mappings were obtained in [4,7,20] if the cost function c satisfies

(**A**) $\forall (x_0, y_0) \in \Omega \times \Omega^*$, the mappings $x \in \overline{\Omega} \to D_y c(x, y_0)$ and $y \in \overline{\Omega}^* \to D_x c(x_0, y)$ are diffeomorphisms onto their ranges.

The regularity of optimal mappings was a more complicated issue. For the quadratic cost function, it reduces to the regularity of the standard Monge–Ampère equation, of which the regularity has been studied by many authors (see for instance [5,6]). For general costs, the regularity was obtained in [26] if the domains satisfy a certain convexity condition, f, g are positive and smooth, and the cost function c satisfies the following structure condition:

(**B**) $\forall x \in \overline{\Omega}$, $y \in \overline{\Omega}^*$, and vectors $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$\sum_{i,j,k,l,p,q,r,s} \xi_i \xi_j \eta_k \eta_l \left(c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}\right) c^{r,k} c^{s,l}(x,y) \geqslant \beta_0 |\xi|^2 |\eta|^2,$$

where β_0 is a positive constant. Loeper [24] showed that the optimal mapping may not be continuous if the condition (**B**) is violated, i.e. when there exist $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$ such that the left hand side is negative. There are many follow-up researches on the regularity, in both the Euclidean space [23,33] and on manifolds [2,12,18,22,25]. See also [31] for recent development.

Monge's mass transfer problem is a prototype of the optimal transportation and the function (1.1) is the natural cost function. Therefore the existence and regularity of optimal mappings for Monge's problem are of particular interest. However this cost function does not satisfy both key conditions, namely the condition (**A**) for the existence and the condition (**B**) for the a priori estimates.

The existence of optimal mappings for Monge's problem has been studied by many researchers and was finally proved in [8,32]. Prior to that, the existence was also obtained in [16] under some assumptions, and obtained in [30], with a gap fixed in [1]. See also [3,9,10] for the existence of optimal mappings when the norm (1.1) is replaced by a more general norm in the Euclidean space. The proofs in [8,32] are very similar: