J. Math. Pures Appl. 102 (2014) 1062–1079

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Affine embeddings and intersections of Cantor sets

De-Jun Feng [a,*], Wen Huang [b], Hui Rao [c]

[a] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
[b] Department of Mathematics, University of Science and Technology of China, Hefei, 230026, PR China
[c] Department of Mathematics, Central China Normal University, Wuhan, 430070, PR China

ARTICLE INFO

Article history:
Received 6 August 2013
Available online 19 March 2014

MSC:
37D35
37C45
28A75

Keywords:
C^1-embedding
Affine embedding
Self-similar sets
Central Cantor sets

ABSTRACT

Let $E, F \subset \mathbb{R}^d$ be two self-similar sets. Under mild conditions, we show that F can be C^1-embedded into E if and only if it can be affinely embedded into E; furthermore if F cannot be affinely embedded into E, then the Hausdorff dimension of the intersection $E \cap f(F)$ is strictly less than that of F for any C^1-diffeomorphism f on $\mathbb{R}^d$. Under certain circumstances, we prove the logarithmic commensurability between the contraction ratios of E and F if F can be affinely embedded into E. As an application, we show that $\dim_H E \cap f(F) < \min\{\dim_H E, \dim_H F\}$ when E is any Cantor-p set and F any Cantor-q set, where $p, q \geqslant 2$ are two integers with $\log p/\log q \notin \mathbb{Q}$. This is related to a conjecture of Furstenberg about the intersections of Cantor sets.

RÉSUMÉ

Soit E et F deux ensembles auto-similaires dans $\mathbb{R}^d$. Sous des hypothèses raisonnables, on montre qu'il existe un plongement C^1 de F dans E si et seulement s'il existe un tel plongement affine ; de plus, s'il n'existe pas de plongement affine de F dans E, alors pour tout difféomorphisme C^1 de $\mathbb{R}^d$ la dimension de Hausdorff de l'intersection $E \cap f(F)$ est strictement inférieure à celle de F. Dans certains cas, on montre que les logarithmes des facteurs de contraction de E et F sont commensurables lorsqu'il existe un plongement affine de F dans E. En application, on montre que $\dim_H E \cap f(F) < \min\{\dim_H E, \dim_H F\}$ quand E est un p-ensemble de Cantor et F est un q-ensemble de Cantor, où p et q sont des nombres entiers $\geqslant 2$ tels que $\log p/\log q \notin \mathbb{Q}$. Ceci est relié à une conjecture de Furstenberg sur les intersections d'ensembles de Cantor.

* Corresponding author.
E-mail addresses: djfeng@math.cuhk.edu.hk (D.-J. Feng), wenh@mail.ustc.edu.cn (W. Huang), hrao@mail.ccnu.edu.cn (H. Rao).

http://dx.doi.org/10.1016/j.matpur.2014.03.003
0021-7824/

1. Introduction

Let A, B be two subsets of $\mathbb{R}^d$. We say that A can be *affinely embedded* into B if $f(A) \subseteq B$ for some affine map $f: \mathbb{R}^d \to \mathbb{R}^d$ of the form $f(x) = Mx + a$, where M is an invertible $d \times d$ matrix and $a \in \mathbb{R}^d$. Similarly, we say that A can be C^1-*embedded* into B if $f(A) \subseteq B$ for some C^1-diffeomorphism f on $\mathbb{R}^d$.

The objective of this paper is to study the relation between C^1-embeddings and affine embeddings for self-similar sets, and to study the necessary conditions under which one self-similar set can be affinely embedded or C^1-embedded into another self-similar set. These questions are motivated from some studies in related areas, including the classification of self-similar subsets of Cantor sets [6,7], the characterization of Lipschitz equivalence and Lipschitz embedding of Cantor sets [5,2], as well as the study of intersections of Cantor sets [8,3] and the geometric rigidity of $\times m$ invariant measures [10].

Before stating our results, we recall some terminologies about self-similar sets. Let $\Phi = \{\phi_i\}_{i=1}^{\ell}$ be a finite family of contractive mappings on $\mathbb{R}^d$. Following Barnsley [1], we say that Φ is an *iterated function system* (IFS) on $\mathbb{R}^d$. Hutchinson [13] showed that there is a unique non-empty compact set $K \subset \mathbb{R}^d$, called the *attractor* of Φ, such that

$$K = \bigcup_{i=1}^{\ell} \phi_i(K).$$

Correspondingly, Φ is called a *generating IFS* of K. One notices that K is a singleton if and only if the mappings ϕ_i, $1 \leqslant i \leqslant \ell$, have the same fixed point. We say that Φ satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set $V \subset \mathbb{R}^d$ such that $\phi_i(V)$, $1 \leqslant i \leqslant \ell$, are pairwise disjoint subsets of V. Similarly, we say that Φ satisfies the *strong separation condition* (SC) if $\phi_i(K)$ are pairwise disjoint subsets of K. The SC always implies the OSC.

A mapping $\phi: \mathbb{R}^d \to \mathbb{R}^d$ is called a *similitude* if ϕ is of the form $\phi(x) = \alpha R(x) + a$ for $x \in \mathbb{R}^d$, where $\alpha > 0$, R is an orthogonal transformation and $a \in \mathbb{R}^d$. When all maps in an IFS Φ are similitudes, the attractor K of Φ is called a *self-similar* set; in this case, the *self-similar dimension* of K is defined as the unique positive number s so that $\sum_{i=1}^{\ell} \rho_i^s = 1$, where ρ_i denotes the contraction ratio of ϕ_i. It is well known [13] that $\dim_H K = s$ if Φ consists of similitudes and satisfies the OSC, here $\dim_H$ denotes the Hausdorff dimension (cf. [4]); the condition of OSC can be further replaced by some significantly weaker separation condition in the case $d = 1$ and $s \leqslant 1$ [11].

In the remaining part of this section, we assume that $\Phi = \{\phi_i\}_{i=1}^{\ell}$ and $\Psi = \{\psi_j\}_{j=1}^{m}$ are two families of contractive similitudes of $\mathbb{R}^d$ of the form

$$\phi_i(x) = \alpha_i R_i(x) + a_i, \qquad \psi_j(x) = \beta_j O_j(x) + b_j, \quad i = 1, \ldots, \ell,\ j = 1, \ldots, m, \tag{1.1}$$

where $0 < \alpha_i, \beta_j < 1$, $a_i, b_j \in \mathbb{R}^d$ and R_i, O_j are orthogonal transformations on $\mathbb{R}^d$. Let E, F be the attractors of Φ and Ψ, respectively. To avoid triviality, we always assume that E, F are not singletons in this paper.

For any real invertible $d \times d$ matrix M, let $\kappa(M)$ denote the *condition number* of M, that is,

$$\kappa(M) = \max\left\{\frac{|Mu|}{|Mv|}\colon\ u, v \in \mathbb{R}^d \text{ with } |u| = |v| = 1\right\}.$$

The first result of this paper is the following:

Theorem 1.1. *Assume that Φ satisfies the OSC, and the Hausdorff dimension of F equals its self-similar dimension. Then F can be C^1-embedded into E if and only if F can be affinely embedded into E. Furthermore if F cannot be affinely embedded into E, then*