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Spreading speed and profile for nonlinear Stefan problems in high space dimensions

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ABSTRACT

We consider nonlinear diffusion problems of the form $u_t = \Delta u + f(u)$ with Stefan type free boundary conditions, where the nonlinear term $f(u)$ is of monostable, bistable or combustion type. Such problems are used as an alternative model (to the corresponding Cauchy problem) to describe the spreading of a biological or chemical species, where the free boundary represents the expanding front. We are interested in its long-time spreading behavior which, by recent results of Du, Matano and Wang [10], is largely determined by radially symmetric solutions. Therefore we will examine the radially symmetric case, where the equation is satisfied in $|x| < h(t)$, with $|x| = h(t)$ the free boundary. We assume that spreading happens, namely $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t, |x|) = 1$. For the case of one space dimension ($N = 1$), Du and Lou [8] proved that $\lim_{t\to\infty} \frac{h(t)}{t} = c^*$ for some $c^* > 0$. Subsequently, sharper estimate of the spreading speed was obtained by the authors of the current paper in [11], in the form that $\lim_{t\to\infty}[h(t) - c^*t] = \hat{H} \in \mathbb{R}^1$. In this paper, we consider the case $N \geq 2$ and show that a logarithmic shifting occurs, namely there exists $c_* > 0$ independent of N such that $\lim_{t\to\infty}[h(t) - c^*t + (N-1)c_* \log t] = \hat{h} \in \mathbb{R}^1$. At the same time, we also obtain a rather clear description of the spreading profile of $u(t, r)$. These results reveal striking differences from the spreading behavior modeled by the corresponding Cauchy problem.

RÉSUMÉ

On considère des problèmes de diffusion non linéaires de la forme $u_t = \Delta u + f(u)$ pour des condtions aux limites à frontière libre du type de Stefan, le terme non linéraire $f(u)$ est du type monostable, ou bistable, ou de combustion. De tels problèmes sont considérés comme modèles alternatifs (au problème de Cauchy) pour décrire la dispersion d'espèces chimiques ou biologiques, la frontière libre représente le front de dispersion. On s'intéresse aux comportements dispersifs en temps long qui, d'après des résultats récents de Du, Matano et Wang [10], sont surtout

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déterminés par des solutions radialement symétriques. Ainsi on examine le cas radialement symétrique pour $|x| < h(t)$, $|x| = h(t)$ est la frontière libre. On suppose que la dispersion se produit, à savoir $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t, |x|) = 1$. Dans le cas unidimensionnel en espace ($N = 1$) Du et Lou [8] ont montré que $\lim_{t\to\infty} \frac{h(t)}{t} = c^*$ pour une valeur de $c^* > 0$. Par la suite une estimation plus précise a été obtenue par les auteurs dans un article [11] sous la forme : $\lim_{t\to\infty}[h(t) - c^*t] = \hat{H} \in \mathbb{R}^1$. Dans cet article on considère les cas $N \geqslant 2$ et on montre qu'il existe une constante $c_* > 0$, indépendante de N, telle que $\lim_{t\to\infty}[h(t) - c^*t + (N-1)c_* \log t] = \hat{h} \in \mathbb{R}^1$. De même on obtient une description assez précise du profil de dispersion $u(t, r)$. Les résultats montrent des différences importantes de la dispersion si on les compare avec ceux obtenus dans le modèle du problème de Cauchy correspondant.

1. Introduction

We are interested in the long-time limit of the spreading speed and profile determined by the following free boundary problem:

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } x \in \Omega(t),\ t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t),\ t > 0, \\ u(0, x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases} \tag{1.1}$$

where $\Omega(t) \subset \mathbb{R}^N$ ($N \geq 2$) is bounded by the free boundary $\Gamma(t)$ (i.e., $\Gamma(t) = \partial\Omega(t)$), with $\Omega(0) = \Omega_0$, which is a bounded domain with smooth boundary $\partial\Omega_0$, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. μ is a given positive constant, and the nonlinearity $f(u)$ is assumed to be of monostable, bistable or combustion type, whose meanings will be made precise below.

When $f(u) \equiv 0$, (1.1) reduces to the classical one-phase Stefan problem, which arises in the study of the melting of ice in contact with water, with $u(t, x)$ representing the temperature of the water. In the setting of (1.1), the water region $\Omega(t)$ is surrounded by ice, and the free boundary $\Gamma(t) = \partial\Omega(t)$ represents the interphase between water and ice. A nonlinear Stefan problem of the form (1.1) may arise if water is replaced by a chemically reactive and heat diffusive liquid surrounded by ice, with $f(u)$ representing the reaction.

Our motivation to study the nonlinear Stefan problem (1.1) arises from the wish to better understand the spreading of a new species, where u is viewed as the density of such a species, and the free boundary represents the spreading front, beyond which the species cannot be observed, i.e., the species has density 0 outside $\Omega(t)$. (The free boundary condition can be deduced from some reasonable ecological assumptions, see [4] for the radially symmetric case.) This is a new point of view to a classical problem.

Indeed, starting from the pioneering works of Fisher [14] and Kolmogorov, Petrovski and Piskunov [20], such a spreading process is usually modeled by the Cauchy problem:

$$\begin{cases} U_t - \Delta U = f(U) & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ U(0, x) = U_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $U_0(x)$ is nonnegative and has nonempty compact support. In such a case, $U(t, x) > 0$ for all $x \in \mathbb{R}^N$ once $t > 0$, but one may specify a certain level set $\Gamma_\delta(t) := \{x : U(t, x) = \delta\}$ as the spreading front, where $\delta > 0$ is small, and $\Omega_\delta(t) := \{x : U(t, x) > \delta\}$ is regarded as the range where the species can be observed. A striking feature of the long time behavior of the front $\Gamma_\delta(t)$ is revealed by Aronson and Weinberger in their classical work [2], namely, when spreading happens (i.e., $U(t, x) \to 1$ as $t \to \infty$), $\Gamma_\delta(t)$ goes to infinity at a constant asymptotic speed in all directions, i.e., for any small $\epsilon > 0$, there exists $T > 0$ so that

$$\Gamma_\delta(t) \subset A_\epsilon(t) := \{x \in \mathbb{R}^N : (c_0 - \epsilon)t \leq |x| \leq (c_0 + \epsilon)t\} \quad \text{for } t \geq T. \tag{1.3}$$