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Lifespan of classical solutions to quasilinear wave equations outside of a star-shaped obstacle in four space dimensions

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ABSTRACT

We study the initial–boundary value problem of quasilinear wave equations outside of a star-shaped obstacle in four space dimensions, in which the nonlinear term under consideration may explicitly depend on the unknown function itself. By some new $L_t^\infty L_x^2$ and weighted $L_{t,x}^2$ estimates for the unknown function itself, together with energy estimates and KSS type estimates, for the quasilinear obstacle problem we obtain a lower bound of the lifespan $T_\varepsilon \geq \exp(\frac{c}{\varepsilon^2})$, which coincides with the sharp lower bound of lifespan estimate for the corresponding Cauchy problem.

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RÉSUMÉ

On étudie l'équation d'onde quasi-linéaire à l'extérieur d'un obstacle étoilé à quatre dimensions de l'espace, dans lequel le terme non linéaire peut dépendre explicitement sur la fonction inconnue. Par de nouvelles estimations de $L_t^\infty L_x^2$ et pondérée type de $L_{t,x}^2$ pour la fonction inconnue ainsi que les estimations de l'énergie et des estimations de type KSS, pour le problème quasi linéaire de l'obstacle en forme d'étoile, on obtient une borne inférieure de la durée de vie $T_\varepsilon \geq \exp(\frac{c}{\varepsilon^2})$ qui coïncide avec la borne inférieure de la durée de vie qui est précise pour le problème de Cauchy correspondant.

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1. Introduction and main result

This paper is devoted to study the lifespan of classical solutions to the following initial–boundary value problem for nonlinear wave equations:

$$\begin{cases} \Box u(t,x) = F(u, \partial u, \partial\nabla u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^4 \backslash \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ t = 0\colon \quad u = \varepsilon f, \quad u_t = \varepsilon g, \quad x \in \mathbb{R}^4 \backslash \mathcal{K}, \end{cases} \tag{1}$$

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where $\Box = \partial_t^2 - \Delta$ is the wave operator, $\varepsilon > 0$ is a small parameter, the obstacle $\mathcal{K} \subset \mathbb{R}^4$ is compact, smooth and strictly star-shaped with respect to the origin, and f, g in (1) belong to $C_c^\infty(\mathbb{R}^4 \backslash \mathcal{K})$. Moreover $(t, x) = (x_0, x_1, x_2, x_3, x_4)$, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}(\alpha = 0, \cdots, 4)$, $\nabla = (\partial_1, \partial_2, \partial_3, \partial_4)$, $\partial = (\partial_0, \nabla)$. Let

$$\widehat{\lambda} = \big(\lambda; (\lambda_i), i = 0, \cdots, 4; (\lambda_{ij}), i, j = 0, \cdots, 4, i + j \geq 1\big). \tag{2}$$

Suppose that in a neighborhood of $\widehat{\lambda} = 0$, say, for $|\widehat{\lambda}| \leq 1$, the nonlinear term $F = F(\widehat{\lambda})$ is a smooth function satisfying

$$F(\widehat{\lambda}) = \mathcal{O}\big(|\widehat{\lambda}|^2\big) \tag{3}$$

and being affine with respect to λ_{ij} $(i, j = 0, \cdots, 4, i + j \geq 1)$.

Our aim is to study the lifespan of classical solutions to (1). By definition, the lifespan T_ε is the supremum of all $T > 0$ such that there exists a classical solution to (1) on $0 \leq t \leq T$, i.e.

$$T_\varepsilon \stackrel{\text{def}}{=} \sup\big\{T > 0 : (1) \text{ has a unique classical solution on } [0, T]\big\}. \tag{4}$$

First, it is needed to illustrate why we consider the case of spatial dimension $n = 4$. For this purpose, we have to review the history on the corresponding Cauchy problem in four space dimensions. In [1], Hörmander considered the following Cauchy problem of nonlinear wave equations in four space dimensions:

$$\begin{cases} \Box u(t, x) = F(u, \partial u, \partial\nabla u), \quad t \geq 0,\ x \in \mathbb{R}^4, \\ t = 0\colon \quad u = \varepsilon f, \quad \partial_t u = \varepsilon g. \end{cases} \tag{5}$$

Here in a neighborhood of $\widehat{\lambda} = 0$, the nonlinear term F is a smooth function with quadratic order with respect to its arguments. $f, g \in C_c^\infty(\mathbb{R}^4)$, and $\varepsilon > 0$ is a small parameter. He proved that if $\partial_u^2 F(0, 0, 0) = 0$, then (5) admits a unique global classical solution. For general F, he got a lower bound of the lifespan $T_\varepsilon \geq \exp(\frac{c}{\varepsilon})$, where c is a positive constant independent of ε. But this result is not sharp. In [2], Li and Zhou showed that Hörmander's estimate can be improved by

$$T_\varepsilon \geq \exp\bigg(\frac{c}{\varepsilon^2}\bigg). \tag{6}$$

Li and Zhou's proof was simplified by Lindblad and Sogge in [3] later. Recently, the sharpness of Li and Zhou's estimate was shown by Takamura and Wakasa in [4] (see also Zhou and Han [5]). They proved that for the following Cauchy problem of semilinear wave equations:

$$\begin{cases} \Box u(t, x) = u^2, \quad t \geq 0,\ x \in \mathbb{R}^4, \\ t = 0\colon \quad u = \varepsilon f, \quad \partial_t u = \varepsilon g, \end{cases} \tag{7}$$

the lifespan of classical solutions admits an upper bound: $T_\varepsilon \leq \exp(\frac{c}{\varepsilon^2})$ for some special functions $f, g \in C_c^\infty(\mathbb{R}^4)$. In fact, when the spatial dimension $n = 4$, the equation in (7) is just corresponding to the critical case of the Strauss conjecture, so it is the most difficult case to be handled. For the Strauss conjecture, we refer the reader to Strauss [6] and the survey article Wang and Yu [7].

The pioneering works by F. John and S. Klainerman open the field of lifespan estimate of classical solutions to the Cauchy problem of nonlinear wave equations. In other spatial dimensions, classical references can be found in John [8–10], John and Klainerman [11], Klainerman [12–17], Christodoulou [18], Hörmander [19,20], Lindblad [21], Li and Chen [22], Li and Zhou [23–25], Alinhac [26–28], etc. Especially, Klainerman's commutative vector field method in [14] offers the basic framework for treating this kind of problem.

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