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# Recovering the isometry type of a Riemannian manifold from local boundary diffraction travel times ☆

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ABSTRACT

We analyze the inverse problem, if a manifold and a Riemannian metric on it can be reconstructed from the sphere data. The sphere data consist of an open set $U \subset \tilde{M}$ and the pairs $(t, \Sigma)$ where $\Sigma \subset U$ is a smooth subset of a generalized metric sphere of radius $t$. This problem is an idealization of a seismic inverse problem, originally formulated by Dix [8], of reconstructing the wave speed inside a domain from boundary measurements associated with the single scattering of waves. In this problem, one considers a domain $\tilde{M}$ with a varying and possibly anisotropic wave speed which we model as a Riemannian metric $g$. For our data, we assume that $\tilde{M}$ contains a dense set of point diffractors and that in a subset $U \subset \tilde{M}$, we can measure the wave fronts of the waves generated by these. The inverse problem we study is to recover the metric $g$ in local coordinates anywhere on a set $M \subset \tilde{M}$ up to an isometry (i.e. we recover the isometry type of $M$). To do this we show that the shape operators related to wave fronts produced by the point diffractors within $\tilde{M}$ satisfy a certain system of differential equations which may be solved along geodesics of the metric. In this way, assuming that we know $g$ as well as the shape operator of the wave fronts in the region $U$, we may recover $g$ in certain coordinate systems (e.g. Riemannian normal coordinates centered at point diffractors). This generalizes the method of Dix to metrics which may depend on all spatial variables and be anisotropic. In particular, the novelty of this solution lies in the fact that it can be used to reconstruct the metric also in the presence of the caustics.



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RÉSUMÉ

On analyse un problème inverse, si une variété riemannienne peut être reconstruite à partir des données sphère. Les données sphère sont constituées d'un ensemble ouvert $U \subset \tilde{M}$ et les paires $(t, \Sigma)$, où $\Sigma \subset U$ set un sous-ensemble lisse d'une sphère métrique généralisée. Ce problème est une idéalisation d'un problème sismique inverse, à l'origine formulé par Dix [8], consistant à reconstruire la vitesse d'onde dans un domaine à partir des mesures aux frontières associées à la dispersion simple des ondes sismiques. On considère un domaine $\tilde{M}$ avec une vitesse d'onde variable et éventuellement anisotrope modélisée par une métrique riemannienne $g$. On suppose que $\tilde{M}$ contient une densité élevée de points diffractants et que dans un sous-ensemble $U \subset \tilde{M}$, correspondant à un domaine contenant les instruments de mesure, on peut mesurer les fronts d'onde de la diffusion simple des ondes diffractées depuis les points diffractants. Le problème inverse étudié consiste à reconstruire la métrique $g$ en coordonnées locales sur l'ensemble $M \subset \tilde{M}$ modulo une isométrie (i.e. on reconstruit le type d'isométrie). Pour ce faire on montre que l'opérateur de forme relatif aux fronts d'onde produits par les points diffractants dans $M$ satisfait un certain système d'équations différentielles qui peut être résolu le long des géodésiques de la métrique. De cette manière, en supposant que l'on connaît $g$ ainsi que l'opérateur de forme des fronts d'onde dans la région $U$, on peut retrouver $g$ dans un certain système de coordonnées (e.g. coordonnées normales riemanniennes centrées aux points diffractants). Ceci généralise la méthode géophysique de Dix à des métriques qui peuvent dépendre de toutes les variables spatiales et être anisotropes. En particulier, la nouveauté de cette solution est de pouvoir être utilisée pour reconstruire la métrique, même en présence de caustiques.



## 1. Introduction: motivation of the problem

We consider a Riemannian manifold, $(M, g)$, of dimension $n$ with boundary $\partial M$. We analyze the inverse problem, originally formulated by Dix [8] in reflection seismology, aimed at reconstructing $g$ from boundary measurements associated with second-order expansions of diffraction travel times. When the waves produced by a source $F$ are modeled by the solution of the wave equation $(\partial_t^2 - \Delta_g)u(x, t) = F(x, t)$ on $(M, g)$, the geodesics $\gamma_{x,\eta}$ on $M$ correspond to the rays following the propagation of singularities by the parametrix corresponding with the wave operator on $(M, g)$ and the metric distance $d(x_1, x_2)$ of the points $x_1, x_2 \in M$ corresponds to the travel time of the waves from the point $x_1$ to the point $x_2$. The phase velocity in this case is given by $\mathrm{v}(x, \alpha) = [\sum_{j,k=1}^n g^{jk}(x)\alpha_j\alpha_k]^{1/2}$, with $\alpha$ denoting the phase or cotangent direction.

Below, we call the sets $\Sigma_{t,y} = \{\gamma_{y,v}(t);\ v \in T_yM,\ \|v\|_g = 1\}$ generalized metric spheres (i.e. the images of the spheres $\{\xi \in T_yM;\ \|\xi\|_g = t\}$ in the tangent space of radius $t$ under the exponential map). We call these sets the generalized spheres in contrast to the metric spheres, that is, the boundaries of metric balls are the sets

$$\partial B(y, t) = \{x \in M;\ d(x, y) = t\}.$$

Since long geodesics on a Riemannian manifold may not be distance minimizing, we have $\partial B(y, t) \subset \Sigma_{t,y}$ where the inclusion may not be equality (see Fig. 1).

The mathematical formulation of Dix' problem is then the following: Assume that

1. We are given an open set $\Gamma \subset \partial M$, the metric tensor $g_{jk}|_\Gamma$ on the boundary, and the normal derivatives $\partial_\nu^p g_{jk}|_\Gamma$ for all $p \in \mathbb{Z}_+$, where $\nu$ is the normal vector of $\partial M$ and $g_{jk}$ is the metric tensor in the boundary normal coordinates.
2. For all $x \in \Gamma$ and $t > 0$, we are given at the point $x$ the second fundamental form of the generalized metric sphere of $(M, g)$ having the center $y_{x,t}$ and radius $t$. Here, $y_{x,t} = \gamma_{x,\nu(x)}(t)$ is the end point of the geodesic that starts from $x$ in the $g$-normal direction $\nu(x)$ to $\partial M$ and has the length $t$.