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# A priori gradient bounds for fully nonlinear parabolic equations and applications to porous medium models

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A B S T R A C T

We prove a priori gradient bounds for classical solutions of the fully nonlinear parabolic equation

$$u_t = F(D^2u, Du, u, x, t).$$

Several applications are given, including the standard porous medium equation.


R É S U M É

On démontre des estimations a priori pour les solutions classiques d'un problème vraiment non linéaire

$$u_t = F(D^2u, Du, u, x, t).$$

On donne ensuite diverses applications telle que l'équation standard du type milieu poreux.


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## 1. Introduction

Consider the general fully nonlinear parabolic problem

$$u_t = F\big(D^2u, Du, u, x, t\big), \quad (x,t) \in Q := \mathbb{T}^d \times (0, +\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}^d. \tag{1.2}$$

To simplify our arguments, we consider the case of the $d$-dimensional torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ for $d \geq 1$. Note that up to the price of technicalities, the case of the whole space $\mathbb{R}^d$ could be also considered. The aim of the paper is to find assumptions on $F$ in order to obtain, for all $t \geq 0$, a gradient bound on $Du$ of the form

$$\big\|Du(\cdot,t)\big\|_\infty \leq \|Du_0\|_\infty. \tag{1.3}$$

The main argument of the proof is to derive the equation in order to get an equation satisfied by $M(t) := \max_{\mathbb{T}^d} \frac{|Du(\cdot,t)|^2}{2}$ and then to prove that $M$ is non-increasing in the viscosity sense.

As an application of our general approach, we prove gradient estimate (1.3) for the weak nonnegative solution of the standard porous medium equation

$$u_t = \Delta u^m, \quad (x,t) \in Q, \tag{1.4}$$

where $1 \leq m \leq 1 + \frac{4}{3+d}$. For such range of $m$, this result is new.

Similar gradient estimates are given for the problem

$$u_t = \Delta G(u), \quad (x,t) \in Q, \tag{1.5}$$

for some class of functions $G$, and for the doubly nonlinear problem

$$u_t = \operatorname{div}\big(\psi\big(u, |Du|^2\big)Du\big), \quad (x,t) \in Q, \tag{1.6}$$

for some class of functions $\psi$. Our estimate will be applied to two specific examples of Eq. (1.6) arising in hydrology (and this was our initial motivation for this work). These two examples are the following equations

$$u_t = \operatorname{div}\left(u(1-u)\frac{Du}{1+|Du|^2}\right), \quad (x,t) \in Q, \tag{1.7}$$

and

$$u_t = \operatorname{div}\big(u(1-u)Du\big), \quad (x,t) \in Q. \tag{1.8}$$

Eq. (1.8) derives from Eq. (1.7) as an approximation for small gradients. In these two equations the function $u$ represents the height of the sharp interface between salt and fresh water in a confined aquifer, see for instance [1,3,7].

### 1.1. Main results

In this subsection, we will present our main results. To this end, we will start by an assumption on the function $F$ appearing in Eq. (1.1). In order to write this assumption, we need to introduce some notation.

For two symmetric matrices $X = (x_{ij})_{1 \leq i,j \leq d}$ and $Y = (y_{ij})_{1 \leq i,j \leq d}$ in $\mathbb{R}^{d\times d}$, we denote by $X : Y$ the inner scalar product $\sum_{i,j=1,..,d} x_{ij}y_{ji} = tr(XY)$. Moreover for $p, q \in \mathbb{R}^d$, we set $(X \cdot p)_i = \sum_{j=1,..,d} x_{ij}p_j$ and $p \cdot q = \sum_{j=1,..,d} p_j q_j$. For later use, we also denote by $tr(X)$ the trace of $X$.